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Electrical Impedance Tomography Reconstruction Through Simulated Annealing with Multi-Stage Partially Evaluated Objective Functions

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Abstract— **The EIT reconstruction problem can be solved as an optimization problem using Simulated Annealing. Different objective functions have already been used: Euclidian distance between the simulated and observed potentials; total least squares error minimization. The objective function was partially evaluated in both methods. In this paper, a new image reconstruction method that combines the best characteristics from both described methods is proposed. The total reconstruction cost is smaller when compared to each method using one objective function alone. A denser reconstruction mesh is used, and a regularization is adopted by adding a new term similar to the first-order Tikhonov functional. The impact of the regularization in the previous two methods is also analyzed.**

I. Introduction

EIT is a diffuse imaging technique for determining the electrical conductivity distribution inside an object from its boundary measurements. A set of electrodes is attached to the object surface, for example, a human body, then electrical current is injected through the electrodes and electrical potential are measured on these electrodes. In EIT it is possible to reconstruct either difference or static images. The difference image modality can be used when changes in the resistivity occur [1]. The reconstruction of static images is substantially more difficult than the difference imaging because it is necessary to have a reference voltage.

The spatial resolution of EIT is not comparable to other imaging techniques such as magnetic resonance, computerized tomography or ultrasonic imaging. However, EIT presents some advantages over these techniques, such as being harmless to the patient, low cost and portable. EIT also has faster time-response characteristics, which enables it to monitor cyclic changes in the living tissues better than conventional imaging modalities. Many researchers have been making continuous efforts in the pursuit of algorithms that are, at the same time, fast and capable of providing images of good spatial resolution.

Martins et al. [6, 7] used the Euclidian distance between the measured electric potentials and the calculated potentials as objective function. Martins and Tsuzuki [9] minimized the total least squares errors. Both methods partially evaluated the objective function. In this paper, it is shown that the characteristics of both methods can be efficiently combined. The total reconstruction cost will be smaller than each of the objective function used alone.

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This paper is structured as follows. Section II reviews current results on EIT reconstruction using Simulated Annealing (SA) with objective functions. Section III proposes a new regularized multi-stage approach where the SA objective function is replaced *during the optimization process*. In section IV some results obtained from physical data are presented. Finally, section V rounds up the paper with the conclusions.

II. Reconstructing EIT images through SA with partial evaluation of the objective function

The forward and inverse problems are briefly explained. Two different objective functions in which are partially evaluated, are described. In the following, both objective functions are combined in just one method.

A. Formulation of the Forward Problem

The flow of electrical current within a conductive thin film, Ω, can be described at any point by the 2D Laplacian equation

$$\nabla(\sigma\nabla\phi) = 0 \tag{1}$$

where σ is the film conductivity and ϕ is the electrical potential. The typical *forward problem* in EIT is given the conductivity distribution σ and the current J injected through boundary electrodes, find the potential distribution ϕ within Ω and in particular the resulting potentials at the measurement electrodes ϕ_m. The frequencies used in EIT are low enough so that the quasi-static approximation holds, and thus one can ignore capacitive and inductive effects. At the boundary, currents are injected through electrodes; thus the current density J_l injected through the l-th electrode is given by (current pattern)

$$\sigma\frac{\partial\phi}{\partial\hat{n}} = J_l \tag{2}$$

where $\hat{n}$ is the external normal versor, and the current density is zero elsewhere at the boundary [2].

B. The Inverse Problem

The inverse problem is formulated as given the injected currents J_l and the potentials at measurement electrodes ϕ_m, find the electrical conductivity distribution σ within Ω. The Laplace equation (1) with Dirichlet and Neumann boundary conditions applied is referred to as the continuum model of the forward problem. If considering only the real part of conductivity, the model is still valid with a unique and strictly positive conductivity function σ [3].

In practice, the potentials on the domain boundary are measured only at a finite number of points, so the Dirichlet

boundary condition is incomplete. For an irregular domain and isotropic media, analytical solution to the Laplace equation (1) with boundary condition (2) is unknown; thus, the partial differential equations were approximated by the FEM, the domain is discretized with triangular linear elements with constant conductivity and both problems, forward and inverse, are solved numerically. When the local element matrices are stated in terms of the global coordinates of the mesh, the global conductivity matrix which includes electrode contact impedance effects, is obtained; then

$$K \cdot \Phi = C \tag{3}$$

where $K(\sigma) \in \mathbb{R}^{s \times s}$ is the conductivity matrix calculated at a given particular distribution σ, Φ is a matrix containing nodal potentials corresponding to each applied current pattern, and $\mathbf{C}$ represents p linearly independent current patterns.

C. Solving the Inverse Problem as an Optimization Problem

One possible approach to the EIT inverse problem is to look at it as an optimization problem. Since, as seen in section II-B, the FEM can be used to solve the forward problem on a simulated domain, one possibility for such objective function is to take the Euclidean distance

$$E(\sigma) = \sqrt{\Sigma |\Phi_m^i - \Phi_c^i(\sigma)|_2} \tag{4}$$

between the measured electric potentials Φ_m^i and the calculated potentials $\Phi_c^i(\sigma)$ for every applied current pattern. Indeed, the minimization of (4) is a classic approach to EIT [4, 5, 6, 7]. In [4] it was pointed that the minimization of (4) using gradient-based algorithms is difficult, since (4) is often ill-conditioned. Herrera et al. [5] avoided the computation of objective function gradients by means of SA and by doing so, managed to reconstruct very accurate conductivity distributions of the body, but at a very high computational cost.

D. SA with Partial Evaluation of the Objective Function

Martins et al. [6, 7] proposed a mitigation of this high computational cost: instead of fully evaluating the objective function E at each SA iteration, an estimate $\tilde{E}$ and upper and lower boundaries E_{max} and E_{min} are obtained. Since SA deals only with variations of the objective function between iterations, those are converted in an estimate and boundaries of variation $\Delta\tilde{E}$, ΔE_{max}, ΔE_{min}. It was shown if in a given iteration $\Delta\tilde{E}$, ΔE_{max}, ΔE_{min} satisfy the following inequalities,

$$P_{err} \geq \min(e^{-\Delta\tilde{E}/kt}, 1) - \min(e^{-\Delta E_{\max}/kt}, 1) \tag{5}$$

$$P_{err} \geq \min(e^{-\Delta E_{\min}/kt}, 1) - \min(e^{-\Delta\tilde{E}/kt}, 1) \tag{6}$$

then the probability of SA at that iteration deviating of an SA with full objective function evaluation is less than P_{err}. Estimates of the objective function were obtained by iteratively solving (3) with Conjugated Gradients (CG) algorithm while obtaining an upper limit on the norm of the error at each CG iteration using a technique described by Meurant [8].

The EIT reconstruction technique that uses the partial evaluation of objective function (4) (as in [6, 7]) will be henceforth labelled as "CG".

E. Total Least Squares Error as an Objective Function

Martins and Tsuzuki [9] pointed that the partial evaluation of (4) could have scalability issues with the mesh density. They proposed a new objective function, also partially evaluated. By taking (3), reordering the variables such that the electrode potentials correspond to the last elements of Φ, one can write

$$\begin{pmatrix} K_{ii} & K_{ic}^T \\ K_{ic} & K_{cc} \end{pmatrix} \begin{pmatrix} \Phi_i \\ \Phi_c \end{pmatrix} = \begin{pmatrix} 0 \\ J_l \end{pmatrix} \tag{7}$$

where Φ_i is the vector of tensions at the internal nodes, Φ_c is the vector of tensions at the electrodes, K_{ii}, K_{ic} and K_{cc} are blocks of the matrix $K(\sigma)$ and J_l is the injected current at the electrodes (the injected current at the internal nodes is zero). By imposing $\Phi_c = \Phi_m$ (that is, the potentials at electrodes of the simulated domain are identical to the measured ones) and allowing an error on (7), then

$$\hat{K}(\sigma)\Phi + \mathbf{e} = \hat{J}_l \tag{8}$$

where

$$\hat{K} = \begin{pmatrix} K_{ii} \\ K_{ic} \end{pmatrix} \quad \hat{J}_l = \begin{pmatrix} -K_{ic}^T \Phi_m \\ J_l - K_{cc}\Phi_m \end{pmatrix} \tag{9}$$

and $\mathbf{e}$ is an error vector added to the reduced system to make it consistent with the replacement $\Phi_c \Rightarrow \Phi_m$. Since the error required tends to zero as Φ_c approaches Φ_m, one could take its minimum, subject to (8), as the measure of consistency between the simulated domain and the physical experiment for a given current pattern. A new objective function is

$$E(\sigma) = \sqrt{\sum_l D^l(\sigma)^2} \tag{10}$$

$$D^l = \min_{\Phi} \left\{ \sqrt{\mathbf{e}^T\mathbf{e}} : \hat{K}(\sigma)\Phi + \mathbf{e} = \hat{J}_l \right\} \tag{11}$$

The minimization problem in (11) is a typical least squares problem. In [9], Martins and Tsuzuki showed how the minimum value can be evaluated interactively with upper and lower boundaries, so (5) and (6) can be used as stopping criteria. One of the limitations of this process is that in order to obtain a good lower boundary for (11) one needs an estimate of the smallest eigenvalue of $\hat{K}^T\hat{K}$. They proposed a technique that can obtain good enough estimates of this eigenvalue *while* calculating lower and upper boundaries for (11) for a single current pattern, although spending much iterations that would be necessary. This "iteration excess" is mitigated by the fact that the eigenvalue obtained when calculating (11) for a single current pattern may be reused for all other patterns.

Another more serious limitation arises from the fact that the outer SA algorithm can make the subproblem of optimizing (11) arbitrarily ill-conditioned by making K_{ic} very close to zero. Under those circumstances, arbitrarily low values of (11) can be obtained. In [9], Martins and Tsuzuki proposed to impose fixed values of conductance on the outer layer of the domain (possibly using the approach in [6], for instance). They also suggested, although did not explore it further, to use a regularization procedure to keep the inner problem

from becoming too much ill-conditioned. The reconstruction technique that uses partial evaluation of objective function (11) will be henceforth labelled "LB" (from the "Lanczos Bidiagonalization" algorithm).

III. Regularization and two-steps reconstruction

The reconstruction processes in [6] and [9] used low density FEM meshes, such that the number of variables in the conductivity parametrization was less than the number of independently observed current values. For denser meshes it is necessary to adopt some form of regularization of the conductivity. The regularization adopted here is a modification to the objective function by adding to it a new term. The new objective function is given by $E^*(\sigma) = E(\sigma) + \alpha\Omega(\sigma)$ where α is a constant and $\Omega(\sigma)$ is a functional. The functional adopted here is the total norm of the gradient, given by

$$\Omega(\sigma) = \int |\nabla\sigma|^2 \, \mathrm{d}x. \tag{12}$$

This is equivalent to the first-order Tikhonov functional [10]. This regularization has also the positive by-product of avoiding the ill-conditioning problems of the approach in [9] without requiring previously known values for the external impedance values.

Although the adoption of regularization allows reconstructions of EIT images using only the LB technique, it is interesting to consider another one because of its drawbacks, specifically the required inner iterations for estimating the smallest eigenvalue of $\hat{K}^T\hat{K}$. This has the effect of creating a fixed minimum number of iterations for the evaluation of the objective function at a single current pattern. This effect is more pronounced at the earlier stages of the optimization process, when few iterations would be required to compare two different solutions. While its overall performance is better, the objective function evaluation on LB is more expensive than objective function evaluation on CG in the earlier stages of the reconstruction.

This leads to the idea of performing a two steps reconstruction: start the reconstruction process using the CG technique, then switch to the LB technique *while keeping the optimization variable state*. If both objective functions have approximately the same convergence trajectory, this should allow a reconstruction method that is cheaper than both approaches proposed in [6] and [9]. Since the optimization variables (the conductivity parameters) are exactly the same in both problems, the only issue remaining is to move the SA parameters (that is, the temperature) from one optimization problem to another, as the objective functions themselves are vastly different. This task is equivalent to "jump starting" a second SA optimization by taking the current variables of an already running optimization process. If the temperature parameter of the second optimization is too high, the produced state will be destroyed, while if the temperature parameter is too low, the new process may become trapped in local minima. Ever since SA was proposed, a relation has been observed between solution acceptance rate and temperature.

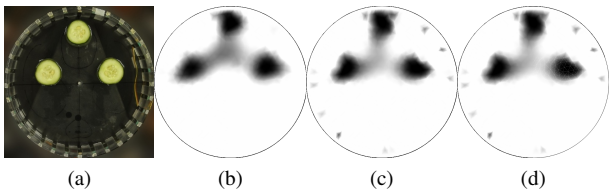

Fig. 1: (a) "Triangle" Phantom and its reconstructions using (b) the approach in [6], (c) the approach in [9] and (d), the aproach presented here. (c) and (d) are virtually undistinguishable.

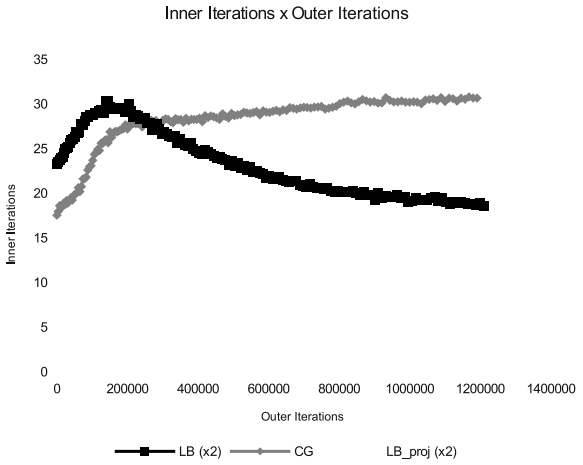

Fig. 2: Inner Iterations $\times$ Outer Iterations, for CG and LB objective functions. LB_proj is an hypothetical evaluation of LB without the cost of estimating the smallest eigenvalue of $\hat{K}^T\hat{K}$.

Since our problem is precisely to find the initial temperature of the second SA optimization, an arbitrary temperature scale is constructed using the progression of acceptance rates both from CG and LB optimizations. With this scale it is possible to convert CG SA temperatures to LB SA temperatures.

IV. Results

For the evaluation of the two-steps approach, the same data for the three cucumbers in triangular shape from [9] was processed, but this time, 1047 nodes and 1868 triangular 1^{st}-order elements (see Fig. 1) were used. Reconstructions were at first performed with the CG technique and then with the LB technique.

The performance of each reconstruction can be seen in Fig. 2 (notice that the cost of a single LB inner iteration is roughly twice the cost of a CG iteration - see [9]). The total cost is roughly proportional to the area bellow the graph. The cost of the reconstruction using LB is only 15% smaller than the reconstruction using CG. The performance of LB is worse than CG in the earlier steps of the reconstruction. This happens because of the need for estimating the smallest eigenvalue of $\hat{K}^T\hat{K}$ (indeed, one can see in Fig. 1 that an hy-

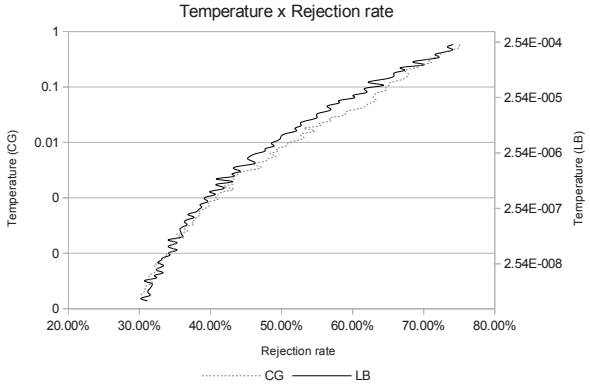

Fig. 3: Temperature $\times$ rejection rate for both CG and LB objective functions.

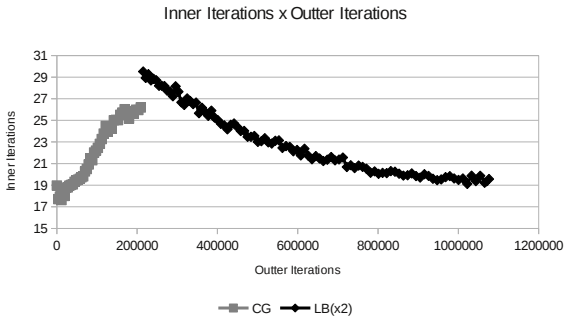

Fig. 4: Inner Iterations $\times$ Outer Iterations, for the two-steps reconstruction.

pothetical LB reconstruction without this cost would almost always outperform CG). As one can see, the reconstructed image through the two-steps technique (Fig. 1d) is virtually indistinguishable from the image obtained from a pure LB reconstruction (Fig. 1c).

To adopt a two-steps reconstruction, it is important to swap objective functions while keeping the SA algorithm roughly in the same state. It is necessary to create a common annealing temperature scale between both objective functions. For that, the solution rejection rate $\times$ temperature for both objective functions is plotted. One can see in Fig. 3, that it is possible to find a temperature scale for which both curves match reasonably well. The swap point was chosen a the point when the rejection rate reaches 50%.

The performance of the two-steps reconstruction can be seen in Fig. 4 and in Table I. As one can see, the objective function swap is performed reasonably well, as the evolution of the LB reconstruction is very similar to that on Fig. 2. The total cost is 22% cheaper than a pure CG reconstruction, and 9% cheaper than a pure LB reconstruction.

The SA algorithm has two phases: first phase where the domain is explored, and second phase where the solution is detailed. It seems that both methods have characteristics associated to one of the SA phases. For a better understanding

TABLE I: Total iterations for the reconstruction methods

Method	Total iterations
CG	31625174
LB ($\times$2)	26597610
two steps	24494677

of such symbioses, further research is necessary.

V. Conclusions

It is proposed here a new approach to solve the EIT inverse problem that combines the best characteristics of the proposals in [6, 7, 9]. This approach uses a two steps SA algorithm that combines two objective functions with partial evaluation. The total reconstruction cost is smaller than when each objective function is used alone. The use of a regularization similar to Tikhonov eliminates the LB potential ill-conditioning. Consequently, the impedance distribution on the outer layer of the domain was calculated.

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