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Uniform Approximation of Gaussian Wavelet for Biomedical Signal Processing in Analog Domain

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Abstract—**Signal processing in analog domain is favorable when power consumption is a critical design constraint. Continuous Wavelet Transform (CWT), which is increasingly being used in characterizing biomedical signals, when implemented in analog domain consumes less power provided the mother wavelet is properly approximated. This paper presents an approximation of Gaussian wavelet by making use of the Uniform approximation. Simulations of the approximated wavelet and the actual wavelet in MATLAB are performed and the results discussed. Simulations show that (i) approximation obtained closely matches the mother wavelet chosen and (ii) a stable approximation which helps in physical realization using any circuit design methodology.**

I. Introduction

Continuous Wavelet Transform (CWT) is used for successful characterization of biological signals. It allows for localization of high frequency signal features in time and vice versa. This time-frequency information is obtained by carrying out the transform at various scales and at various instants of time. In case of ECG, CWT is extensively used to evaluate arrhythmia.

Biomedical signal processing in point of care health technologies calls for low power implementations of circuits. The wavelet transform computed in digital domain is typically implemented in a DSP and is called a Discrete Wavelet Transform (DWT). Low power implementations of the wavelet transform can be obtained by computing CWT. This is because the CWT can be obtained by designing an analog filter whose impulse response closely matches the mother wavelet chosen. This analog filter when implemented using low power techniques helps in using the CWT on portable devices. Particularly for ECG signal analysis, CWT is better suited than DWT because the coefficients of CWT are translation invariant [1].

This paper presents an approximation of the Gaussian wavelet, which is a favorite choice for ECG signal feature extraction. The approximation is based on uniform approximation of transcendental functions. Section II gives a brief overview of the approximation methods presented in literature. It also details how the uniform approximation technique is used to obtain approximation of the Gaussian wavelet.

Section III provides the performance analysis of the obtained Gaussian wavelet transfer function with the Gaussian wavelet function provided in wavelet toolbox of MATLAB, at different scales and time shifts.

II. Uniform approximation

A. Literature Survey

Approximations of wavelets and their low power implementations are presented in [2-7]. Dynamic translinear technique has been used to come up with low power implementations of CWT. In [2-4] Padé approximation has been used to arrive at a rational transfer function. The main advantage of the Padé approximation is the simplicity with which a unique solution can be obtained. The L_2 approximation technique described in [6-7] has been found superior to the Padé approximation but computationally more intensive. One drawback in L_2 based approximation is the choice of the starting point of the approximation, which holds the key for obtaining optimum transfer function. Work reported in [8] suggests a systematic methodology to obtain a starting point for L_2 based approximation. One major difference between the Padé approximation and L_2 based approximation is the domain in which the approximation is arrived at. The Padé approximation is arrived at in the Laplace domain whereas the L_2 based approximation is obtained predominantly in the time domain. However, L_2 based approximation can also be used in the Laplace domain. In the work reported in [8], a rational approximation of the Mexican hat wavelet has been given using McLauren series and is also arrived at in the Laplace domain.

The design procedure illustrated in the above work follows more or less the same overall methodology. Firstly, a suitable mother wavelet is selected depending on the features of the input signal that are to be extracted. Secondly an approximation technique is followed which approximates the chosen mother wavelet closely. Then the scales and translations at which the CWT needs to be calculated are arrived at. Finally it is made sure that the mathematical approximations arrived at for all the pre determined scales and translations are stable so that they are physically realizable. However the order of the transfer function of approximation depends on the tradeoff between accuracy and design criteria. Higher the order of the approximation, better the accuracy but it is limited by practical criteria like chip area, power consumption etc.

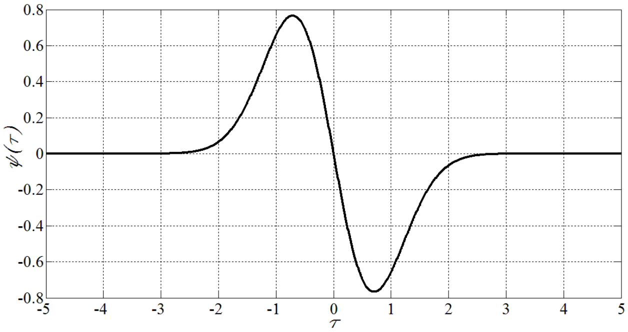

Figure 1. The Gaussian Wavelet as described in (3)

B. *Wavelet Transform*

The CWT of a signal $x(t)$ is given by the equation below [9].

$$W(t,\sigma) = \frac{1}{\sqrt{\sigma}} \int_{-\infty}^{+\infty} x(t)\psi\left(\frac{\tau-t}{\sigma}\right) d\tau \qquad (1)$$

On observation of the above equation, we can say that the wavelet transform of a signal $x(t)$ can be obtained by convolution of the signal $x(t)$ with a function whose impulse response is

$$h(t) = \frac{1}{\sqrt{\sigma}}\psi\left(\frac{-\tau}{\sigma}\right) \qquad (2)$$

If a signal $x(t)$ is passed through a linear system H(s), it is convoluted with the impulse response of H(s). So, a given transfer function H(s) approximates the wavelet, if its impulse response is as given by eqn. (2).

The wavelet of interest in this paper is the Gaussian Wavelet which is given by

$$\psi(t) = -2\left(\frac{2}{\pi}\right)^{\frac{1}{4}} e^{-t^2} t \qquad (3)$$

The Gaussian Wavelet belongs to a family of the Hermitian Wavelets. It is the first derivative of the Gaussian probability distribution function and is illustrated in figure 1. When we attempt to find the Laplace Transform of the Gaussian Wavelet, we obtain integrals which are transcendental and need to be approximated. The transcendental functions usually have series expansions at several points. The series expansions of most of these transcendental functions can be obtained at points like x=0 and x=∞ easily than other points. Expansions at only one of these points are usually not uniform across the interval [0,∞]. Several methods like Lagrange Interpolation and Chebyshev Polynomials do not provide uniform approximations. This calls for a method which can give uniform approximation.

C. *Uniform Approximation*

This section details the Uniform Approximation method used in [10].

The Uniform approximation requires the knowledge of series expansion of a given function f(x) at more than one point, including infinity.

A function $f(x)$ which has series expansions at x=0 and x=∞ and which is finite in the interval (0, +∞) can be approximated as

$$f(x) \approx \frac{p(x)}{q(x)} = \frac{p_0 + p_1x + p_2x^2 + \ldots + p_nx^n}{q_0 + q_1x + q_2x^2 + \ldots + q_nx^n} \qquad (4)$$

The coefficients p_i and q_i are such that the above equation matches with the series expansion of f(x) at both x=0 and at x=∞. This is a problem similar to the Hermite – Padé interpolation with two anchor points. The above equation is approximated such that (4) has correct expansions at both x=0 and x=∞. Since the value of (4) at infinity is p_n/q_n, we can choose $p_n = q_n = 1$. The remaining unknowns, after the values of p_n and q_n are fixed, are 2n. Equations (5) and(6) yield a system of (m+k-1) equations. We choose m and k such that the number of linear equations that can be written from the equations (5) and (6) equals the number of unknowns, that is, 2n. The polynomials a(x) and b(x) are known from the series expansions of the function f(x) at x=0 and at x=∞ respectively.

$$p(x) - q(x)a(x) = O(x^m) \qquad (5)$$

$$\frac{p(x)}{x^n} - \frac{q(x)}{x^n} b(x) = O(x^k) \qquad (6)$$

D. *Uniform Approximation on Gaussian Wavelet*

While trying to find out the Laplace Transform of Gaussian derivatives, we usually come across the complementary error function. The complementary error function is given by the equation below:

$$erfc(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} g(x) \qquad (7)$$

The function $g(x)$ can be written as series expansion at x=0 and x=∞, as given below in equations (8) and (9)

$$At\ x = 0, g(x) = \sqrt{\pi}x - 2x^2 + \sqrt{\pi}x^3 - \frac{4}{3}x^4 + O(x^5) \qquad (8)$$

$$At\ x = \infty, g(x) = 1 - \frac{1}{2x^2} + \frac{3}{4x^4} + O(x^{-6}) \qquad (9)$$

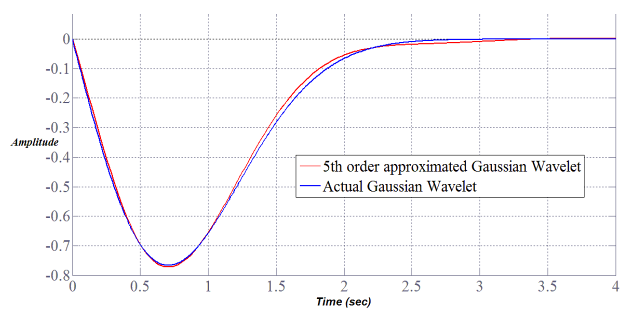

Figure 2. 5th order approximated Gaussian Wavelet with t=0, compared to the actual Gaussian Wavelet

Using the above expansions for $g(x)$, the Laplace transform of Gaussian Wavelet can be approximated by the Uniform approximation method. A simple second order approximation of the Gaussian Wavelet in eqn. (3) is given by eqn. (10) given below:

$$\psi(x) \approx \frac{1.14x^2 + 3.44x}{1.14x^2 + 3.44x + 4} \quad (10)$$

Figure 2 illustrates the 5th order approximation of the un-shifted Gaussian wavelet with the actual wavelet given in equation (11). As can be seen from the figure, the approximated Gaussian Wavelet closely matches the actual un-shifted Gaussian Wavelet. The plot is presented from time t=0 because the system has to be causal if it has to be implemented using circuitry. The higher order approximations of the filter yield better accuracy, but the order is limited by practical issues.

The CWT has to be computed at several scales and translations of the given mother wavelet. The more the number of these scales and translations more is the circuitry area involved and hence more is the power consumed. Typically transfer functions for several scales and translations are calculated and then they are implemented as a filter bank at the input stage. Figure 3 shows different orders of approximation of the Gaussian Wavelet done using the uniform approximation at different scales and translations. Equation (12) gives the transfer function of the approximated Gaussian Wavelet at a scale of 0.5 and shifted by 1sec and eqn. (13) gives the transfer function of the approximated Gaussian Wavelet at a scale of 1.0 and shifted by 2sec.

$$H(s) = \frac{0.02738s^5 + 0.2512s^4 + 1.205s^3 + 3.18s^2 + 3.854s}{0.03125s^5 + 0.2868s^4 + 1.43s^3 + 4.207s^2 + 6.905s + 4.96} \quad (11)$$

III. Results

A. Procedure

The proposed approximation technique has been tested with the Gaussian Wavelet present in MATLAB. The procedure followed is similar to the one followed in [8] and is outlined below:

1. ECG data of required duration is obtained from the MIT –BH database, into a variable in MATLAB. This forms the input signal.
2. The impulse response of the approximated transfer function (at a particular scale and translation) is obtained.
3. The input signal obtained in step 1 is then convolved with the impulse response obtained in step 2 and plotted
4. Then the Gaussian Wavelet (at the same scale and translation) inbuilt in MATLAB is convolved with the same input ECG signal and plotted.
5. The two plots are then superimposed and compared

$$Scale = 0.5 \; t = 1;$$

$$H(s) = \frac{0.354s^2 + 4.041s}{0.03562s^4 + 0.5763s^3 + 3.551s^2 + 9.593s + 12.25} \quad (12)$$

$$Scale = 1 \; t = 2;$$

$$H(s) = \frac{2.097e6\, s^5 + 1.086e7\, s^4 + 4.719e7s^3 + 1.084e8s^3 + 1.267e8\, s}{1.18e6\, s^7 + 6.504e6\, s^6 + 3.041e7\, s^5 + 8.384e7\, s^4 + 1.542e8\, s^3 + 1.757e8\, s^2 + 1.152e8\, s + 3.622e7} \quad (13)$$

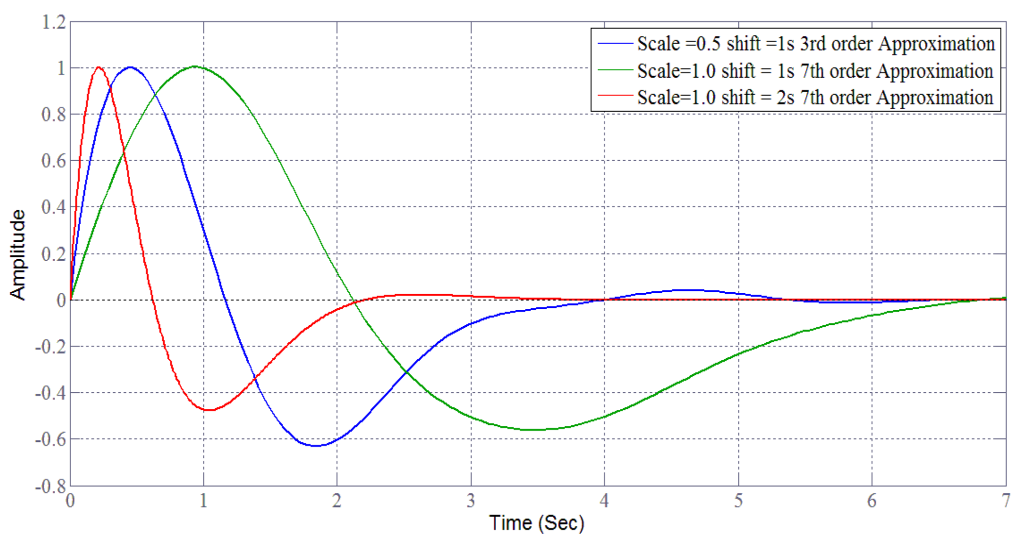

Figure 3. The approximated Gaussian Wavelets at different scales and time shifts

B. Analysis

Graphs obtained following the above procedure are plotted in the figure 4. The first part shows a comparison of plot obtained with MATLAB and plot obtained from the approximated (5^{th} order) transfer function when there is no shift and the scale is 1. The ECG data is of 10 second duration. We can examine that the plot from the approximated wavelet closely follows the plot obtained from the MATLAB.

The second part of figure 4 shows the comparison of plot obtained with MATLAB with plot obtained from the approximated (7^{th} order) transfer when the shift is 2 and scale is 1. In this case, a ECG signal of one minute duration is considered. The plot obtained from the approximated wavelet closely follows the plot obtained from MATLAB. Even better results can be obtained by opting for a higher order approximation.

Since in both the above cases plots are closely matched, the actual performance of the approximated Gaussian Wavelet at different scales and translations would yield results close to the coefficients that MATLAB would produce. The obtained approximated transfer functions are also stable and their stability has been checked by the pole – zero plot given in figure 5. As seen, all the poles are present on the left half plane, which indicates clear stability. This enables the obtained transfer function to be realized in any of the low power implementations present in the literature.

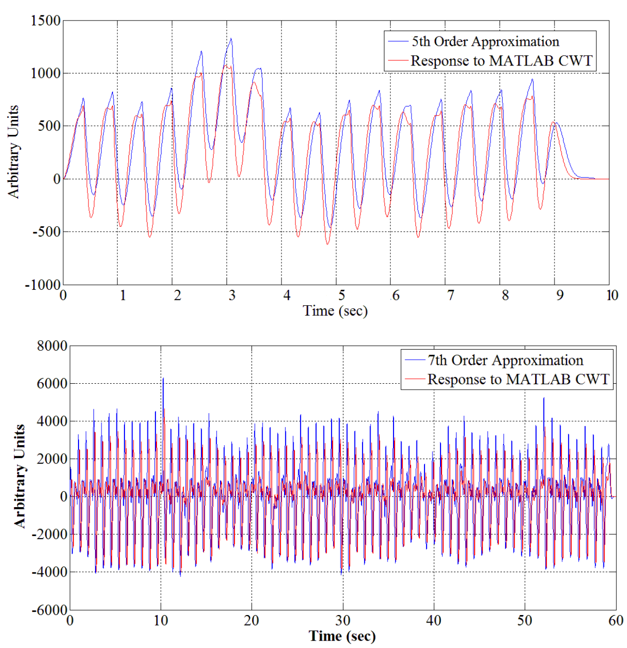

Figure 4. The comparison of approximated wavelet with the MATLAB CWT

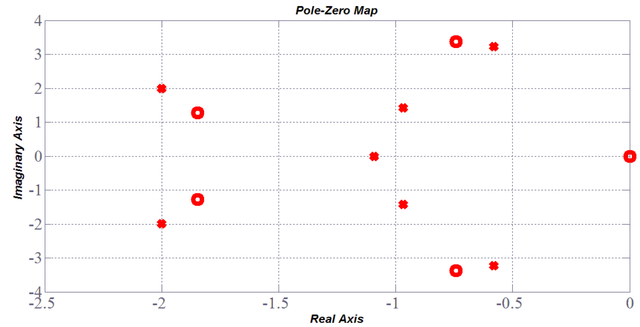

Figure 5. Pole Zero plot of the 7^{th} order approximated wavelet

IV. CONCLUSION

Approximation of Gaussian Wavelet has been presented in this paper. To obtain the approximation in the entire region $(0,+\infty)$ uniformly, Uniform approximation method has been followed and the transfer functions of the Gaussian wavelet at several scales and time shifts have been arrived at. Results show that the approximation closely follows the CWT in MATLAB. The transfer functions obtained have all been stable and hence physically realizable.

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