34th Annual International Conference of the IEEE EMBS
San Diego, California USA, 28 August - 1 September, 2012

Subspace Identification of Hammerstein Systems Using B-Splines

K. Jalaleddini, Student Member, IEEE, D. T. Westwick, Member IEEE, and R. E. Kearney, Fellow, IEEE

Abstract— **This paper presents an algorithm for the identification of Hammerstein cascades with hard nonlinearities. The nonlinearity of the cascade is described using a B-spline basis with fixed knot locations; the linear dynamics are described using a state-space model. The algorithm automatically estimates both the order of the linear system and the number and locations of the knots used to characterize the nonlinearity. Therefore, it significantly reduces the *a priori* knowledge about the underlying system required for identification. A simulation study on a model of reflex stiffness shows that the new method estimates the nonlinearity accurately in the presence of output noise.**

I. INTRODUCTION

The Hammerstein structure consists of a zero memory static nonlinearity followed by a linear dynamic system as illustrated in Fig. 1 [1], [2]. Biological examples include the reflex stiffness of the human ankle joint and the mechanical behavior of lung tissue [3], [4]. Therefore, the accurate identification of Hammerstein systems is an important problem.

Subspace methods are a well-developed set of tools for the identification of linear systems. They represent a linear system by a state-space model that can be estimated with no *a priori* knowledge about the system order [5], [6].

Recently, we developed a subspace algorithm for the identification of Hammerstein cascades that uses the framework proposed in [7] to estimate the parameters corresponding to the nonlinearity separately from those of the linear state-space model. The algorithm models the nonlinearity with an orthogonal Tchebychev polynomial, and separates the parameters into two sets: one corresponding to the static nonlinearity and the second to the state-space model. The output is a linear function of each parameter set provided the other set is held constant. Consequently, an iterative least-squares procedure can be used to find the optimum nonlinear and linear component parameters [8].

We assessed the performance of this algorithm using a small signal model of ankle stretch reflex stiffness where we modeled the nonlinearity with a half-wave rectifier (threshold) and the linear component with a second-order low-pass filter. We demonstrated that the algorithm could distinguish changes in threshold from those in the linear component gain [9].

K. Jalaleddini is with the Department of Biomedical Engineering, McGill University, 3775 University, Montréal, Québec H3A 2B4, Canada. seyed.jalaleddini@mail.mcgill.ca.

D. T. Westwick is with the Department of Electrical and Computer Engineering, University of Calgary, 2500 University Drive NW, Calgary, AB T2N 1N4, Canada. dwestwic@ucalgary.ca.

R. E. Kearney is with the Department of Biomedical Engineering, McGill University, 3775 University, Montréal, Québec H3A 2B4, Canada. kearney@mcgill.ca.

This work has been supported by FQRNT and CIHR.

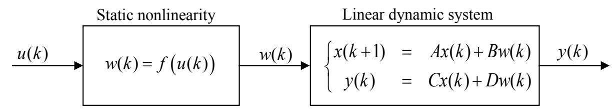

Fig. 1. Hammerstein model as a cascade of nonlinear-linear block.

A more general model for the reflex stiffness of one muscle would include both threshold and saturation behaviors [10]. Moreover, joints are controlled by multiple muscles which can be expected to have different thresholds and saturations. This could lead to nonlinearities with sharp changes in slopes. The presence/absence of these corner points could be significant in interpreting the underlying physiology [11]. Pilot experimental results from our laboratory confirm that the reflex nonlinearity is more complex than a simple half-wave rectifier [9].

Such hard nonlinearities are difficult to model using finite-order polynomials due to problems with oscillations and instability. Consequently, it is difficult to accurately estimate the corner points when using a Tchebychev expansion to describe the nonlinearity. One solution to this problem is to represent the nonlinearities using splines as in [12].

The contribution of this paper is twofold. First, we develop a subspace identification method for Hammerstein cascades using splines. Splines have been used for Hammerstein identification previously, but the linear component was described in terms of its *impulse response function* (IRF) [12]. Replacing the IRF with a state-space model can reduce the number of unknown parameters dramatically - especially for systems with large memory. Therefore, state-space identification should be more robust in presence of noise.

Second, in our spline formulation, we show how to choose number of knots and their locations to describe the static nonlinearity parsimoniously. This is significant since the proper choice of the nonlinearity is not well understood and is usually based on trial and error.

The paper is organized as follows. Section II reviews the B-spline basis functions, formulates the problem and describes the algorithm. Section III presents the results of a simulation study that evaluates the performance of the new algorithm and compares it to our previous method. Section IV provides a summary and some concluding remarks.

II. THEORY

A. B-Spline

A k-th order B-spline is defined by a set of knot points where the output between each pair of knots is given by a

$(k-1)$-th order polynomial. The first $(k-2)$ derivatives of the spline are continuous at the knot locations [13]. If the knot sequence $\Lambda = \{\lambda_1,\ \lambda_2,\ \cdots, \lambda_{n+k}\}^T$ is as follows:

$$\begin{aligned} \lambda_1 = \cdots = \lambda_k = L_1 < \lambda_{k+1} \le \cdots \le \lambda_n < \\ < L_2 = \lambda_{n+1} = \cdots = \lambda_{n+k} \end{aligned} \tag{1}$$

where L_1 and L_2 are the minimum and maximum of the nonlinearity's input. Then, the spline's output w is defined as:

$$w = \sum_{j=1}^{n} S_j^{\{k\}}(u)\alpha_j \tag{2}$$

where α is the set of coefficients of the B-spline $\alpha = [\alpha_1, \cdots, \alpha_n]^T$ and $S_j^{\{k\}}$ is the sequence of normalized B-splines of order k with respect to the knot sequence Λ and is derived from the following recursive equation:

$$S_j^{\{1\}}(u) = \begin{cases} 1 & \text{if } \lambda_j \le u < \lambda_{j+1} \\ 0 & \text{otherwise} \end{cases} \tag{3a}$$

$$S_j^{\{k\}}(u) = p_j^{\{k\}}(u) S_j^{\{k-1\}}(u) + \left(1 - p_{j+1}^{\{k\}}(u)\right) S_{j+1}^{\{k-1\}}(u) \tag{3b}$$

$$p_j^{\{k\}}(u) = \begin{cases} \frac{u-\lambda_j}{\lambda_{j+k-1}-\lambda_j} & \text{if } \lambda_j < \lambda_{j+k-1} \\ 0 & \text{otherwise} \end{cases} \tag{3c}$$

Now, the output of the nonlinearity based on this approximation is:

$$W = \boldsymbol{S}\alpha \tag{4}$$

where, W is the sampled vector of the output of the nonlinearity $W = [w(1), \cdots, w(N)]^T$ and $\boldsymbol{S}$ is the observation matrix defined as follows:

$$\boldsymbol{S} = \begin{bmatrix} S_1^{\{k\}}(u(1)) & \cdots & S_n^{\{k\}}(u(1)) \\ S_1^{\{k\}}(u(2)) & \cdots & S_n^{\{k\}}(u(2)) \\ \vdots & & \vdots \\ S_1^{\{k\}}(u(N)) & \cdots & S_n^{\{k\}}(u(N)) \end{bmatrix} \tag{5}$$

B. Hammerstein Formulation

Consider the *single input single output* SISO Hammerstein system shown in Fig. 1. Assume that the order of the linear system is m and the elements of the B and D state-space matrices are $B = [b_1, \cdots, b_m]^T$ and $D = [d]$. Transform this SISO nonlinear cascade to a *multi input single output* MISO linear system whose n inputs are the outputs of the constructed spline basis functions, i.e., $U(k) = \left[S_1^{\{k\}}(u(k), \Lambda)\ \cdots\ S_n^{\{k\}}(u(k), \Lambda)\right]^T$. The resulting MISO state-space model is:

$$\begin{cases} x(k+1) & = Ax(k) + B_\alpha U(k) \\ y(k) & = Cx(k) + D_\alpha U(k) \end{cases} \tag{6}$$

where, $x(k)$ is the state vector while, A and C are the linear system state-space matrices. The elements of B_α and D_α are given by:

$$B_\alpha = \begin{bmatrix} b_1\alpha_1 & \cdots & b_1\alpha_n \\ \vdots & \ddots & \vdots \\ b_m\alpha_1 & \cdots & b_m\alpha_n \end{bmatrix} \tag{7}$$

$$D_\alpha = \begin{bmatrix} d\alpha_1 & \cdots & d\alpha_n \end{bmatrix} \tag{8}$$

The measured output $\tilde{y}(k)$ is contaminated with noise:

$$\tilde{y}(k) = y(k) + n(k) \tag{9}$$

If the state-space matrices A and C are known, the output of the Hammerstein system is given by [8], [14]:

$$\begin{aligned} \tilde{y}(k) = \left[\sum_{\tau=0}^{k-1} U^T(\tau) \otimes CA^{k-1-\tau}\right] \text{vec}(B_\alpha) + U^T(k)\text{vec}(D_\alpha) \\ + n(k) \end{aligned} \tag{10}$$

where $\otimes$ is the Kronecker product. Rewriting (10) in a matrix format gives:

$$\begin{aligned} \tilde{Y} = \Psi\left[\alpha_1 b_1\ \cdots\ \alpha_1 b_m\ \cdots\ \alpha_n b_1\ \cdots\ \alpha_n b_m\ \alpha_1 d\ \cdots\ \alpha_n d\right]^T \\ + N \end{aligned} \tag{11}$$

where Ψ is the observation matrix defined using the input signal as well as A and C according to (10). This relation shows that the unknown parameters comprise two sets: α which contains the coefficients of the spline, and $\theta_{bd} = [b_1, \cdots, b_m, d]^T$ which contains the state-space elements.

C. Identification Algorithm

Step 1: Assume the knot sequence Λ, $\lambda_1 = \cdots = \lambda_k = \min(u(k))$ and $\lambda_{n+1} = \cdots = \lambda_{n+k} = \max(u(k))$ where $\lambda_{k+1}, \cdots, \lambda_n$ are equally spaced across the input signal range with the resolution of $\frac{\max(u(k))-\min(u(k))}{n-k}$.

Step 2: Construct the B-spline basis expansion (5) of the input signal using the knot sequence Λ.

Step 3: Use the MOESP algorithm, described in [5], to estimate the A and C matrices of the linear state-space model of (6) using the constructed input signal (U(k)) and noisy output (9).

Step 3: Initialize the coefficients set $\alpha = [1, \cdots, 1]^T_{n\times 1}$.

Step 4: Construct the matrix Ψ_α:

$$\Psi_\alpha = \Psi \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \hat{\alpha}_1 & \cdots & \hat{\alpha}_n \\ 0 & \cdots & \hat{\alpha}_1 & \cdots & 0 & \cdots & \hat{\alpha}_n & 0 & \cdots & 0 \\ \vdots & \iddots & \vdots & & \vdots & \iddots & \vdots & \vdots & & \vdots \\ \hat{\alpha}_1 & \cdots & 0 & \cdots & \hat{\alpha}_n & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}^T \tag{12}$$

Estimate θ_{bd} by solving the least-squares problem: $Y = \Psi_\alpha \theta_{bd}$

Step 5: Construct the matrix Ψ_{bd}:

$$\Psi_{bd} = \Psi \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_m & 0 & \cdots & 0 \\ 0 & b_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_m \\ d & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & d \end{bmatrix} \tag{13}$$

Estimate α by solving the least-squares problem: $Y = \Psi_{bd}\alpha$.

Step 6: Compute the *sum of squared errors* SSE for the model and compare it to that from the previous iteration. Go to step 7 if there is not a significant decrease. Otherwise, go to Step 4.

Step 7: Sort the knot points as follows. Recall that in a k^{th} order spline, the $(k-1)^{\text{th}}$ derivative is discontinuous at the knot locations. If the $(k-1)^{\text{th}}$ derivative is discontinuous at a knot location, that knot is active and contributes to the characterization of the static nonlinearity. If the $(k-1)^{\text{th}}$ derivative is continuous, that knot does not actively contribute in characterization of the static nonlinearity [15]. In the presence of output noise, however, the spline coefficient estimation is not perfect and small discontinuities in the $(k-1)^{\text{th}}$ derivative may be observed at inactive knots. Consequently, we sort the knots according to the amount of discontinuity in the $(k-1)^{\text{th}}$ derivative which can be simply measured from the $(k-2)^{\text{th}}$ derivative at knot locations.

Step 8 Iteratively, identify the system by adding knots according to the order of the sorted sequence of Step 7. Calculate the *mean squared error* (MSE) at each identification. Stop adding knots when no significant improvement in MSE is observed.

III. Simulation Results

We assessed the performance of the algorithm using a small signal model of ankle stretch reflex stiffness. The input to this system is the angular velocity of the ankle joint and the output is the reflex torque. This system was modeled as a Hammerstein system consisting of a half wave rectifier followed by a second-order low pass filter [3], [16].

More recent work has demonstrated that in the human ankle, the threshold is not fixed at zero [1] but changes with the background torque level [9]. Moreover, there is also experimental evidence for a saturation nonlinearity. Furthermore, several muscles, presumably with different thresholds, interact to generate the overall reflex response. Therefore, we considered a more general nonlinearity model consisting of a threshold, an intermediate change of slope, and a saturation as shown in Fig. 2. This type of nonlinearity models three experimental phenomena: (a) the strong unidirectional rate sensitivity (t_1), (b) activation of a set of new muscle fibers (t_2) and (c) the saturation of the response at high velocities (t_3). Consequently, the nonlinearity has three corner points which were set to $t_1 = -0.4,\ t_2 = 0,\ t_3 = 0.4$. We modeled the linear system as a second-order low-pass filter:

$$G(s) = \frac{G_r \omega_n^2}{s^2 + 2s\zeta\omega_n + \omega_n^2} \tag{14}$$

The parameters of the linear element were chosen to be similar to those found experimentally ($G_r = 1,\ \omega_n = 55,\ \zeta = 2.2$) [16].

The input angular joint velocity signal was a uniform random number between -3 and 3 rad/s. We simulated the input and output signals at 1000 Hz for 60s. A realization of white Gaussian noise was added to the output to generate a *signal to noise ratio* (SNR) of 10 dB.

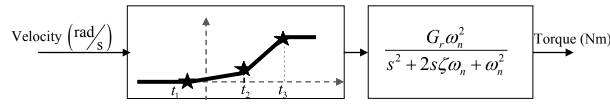

Fig. 2. Hammerstein model of reflex stiffness.

We identified that system from the simulated data using an initial spline of order 2 with 34 knots equally spaced in the range of input.

Fig. 3(A) shows the derivative of the nonlinearity estimated with 34 knots after step 6 of the algorithm. It is evident that the derivative is discontinuous at some knot locations but not others. To separate knots whose discontinuities were not significant from those with significant discontinuities, we sorted the knots by the value of their second derivatives as shown in Fig. 3(B). Fig. 3(C) shows that after selection of the first five knots, the MSE between the predicted output of the Hammerstein cascade and clean output converged to a small number. This indicates that only the first five knots were important and adding more knots would not significantly improve the identification. Consequently, we consider only the first five important knots as active ones.

We identified the system once again using only the active knots and also compared the result with our previous algorithm in [8] which used an 8-th order Tchebychev polynomial with a subspace identification approach. Fig. 4(A) shows the results. It is evident that the spline was more accurate despite having fewer parameters than the polynomial. Moreover, the transition points, which were not easily identified in the Tchebychev polynomial, were clearly evident with splines. The *variance accounted for* (VAF) of the estimated output compared to the clean output using spline was higher than Tchebychev: 99.97% for B-spline and 97.62% for Tchebychev. Fig. 4(B) shows that the frequency response of the identified linear dynamic using B-spline and Tchebychev matched the true system accurately.

IV. Discussion

An identification algorithm was developed for Hammerstein cascade systems. The algorithm uses a subspace approach and is useful for systems with hard nonlinearities. It models the nonlinear element with a B-spline and the linear element with a state-space model. It then transforms a SISO Hammerstein system to a MISO linear system.

Simulation results of a model of ankle reflex stiffness show that the new method provided more accurate estimates of the nonlinearity than our previous subspace method and could successfully detect sharp corner points. This improvement should make possible a better understanding of the underlying physiological information.

The new method requires minimal *a priori* information. The method uses the MOESP subspace algorithm to estimate the A and C state-space matrices. Prior to the identification of these matrices, MOESP estimates the order of the linear system. Second, the method determines the minimal number

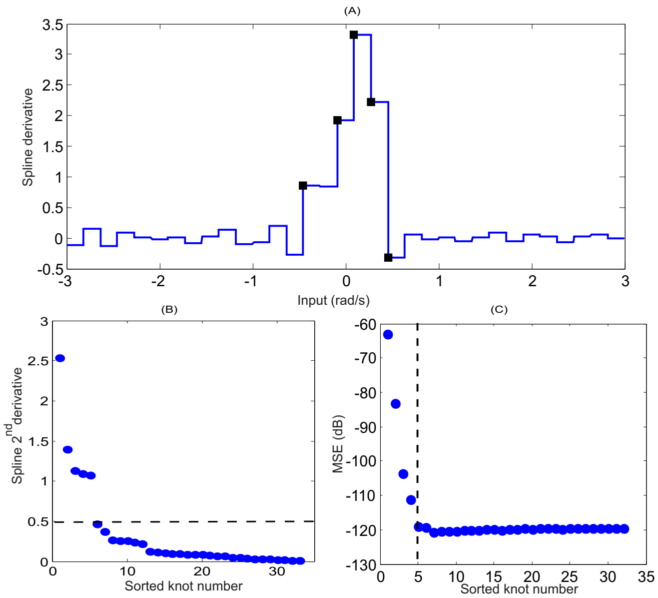

Fig. 3. Selection of active knots: (A) first derivative of the estimated spline; (B) sorted knots according to the second derivative of the spline; (C) MSE according to the sorted knot sequence.

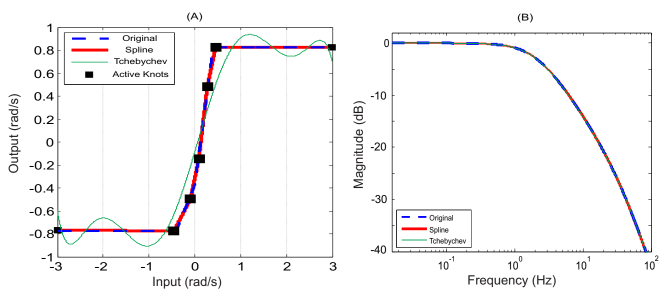

Fig. 4. Identified Hammerstein system: (A) Static nonlinearity, spline using only active knots superimposed on the 8-th order Tchebychev approximation; (B) Identified linear system frequency response.

of knots and their locations required to represent the static nonlinearity.

Another advantage of the method is that it does not require the use of Gaussian inputs. It uses an over-parameterized MISO model and so does not rely on Busgang's theorem and therefore does not require a Gaussian distribution for the input signal. This is useful for experiments where Gaussian inputs cannot be used or generated, such as studies of reflex stiffness where a PRBS input signal is often used for identification. It is also advantageous to use uniformly distributed inputs in Hammerstein identification, since the input can equally excite all regions in the nonlinearity [17].

The knot locations used for the parsimonious model were a subset of those used for the initial segmentation. Consequently, the estimation of corner point locations in the hard nonlinearity is limited to the resolution of segmentation, i.e., location of knots. For instance, in the simulation study, the input range was between -3 to 3 rad/s and we used 34 knots. Therefore, the resolution of corner point estimation is ± 0.09 rad/s. One way to increase the estimation accuracy of the corner points would be to use methods that consider variable knot location. However, for variable knot locations, the problem is highly nonlinear [18], [12]. Therefore, nonlinear optimization techniques need to be used to find the optimum knot location. It is known that if the initial condition of a nonlinear optimization problem is set properly, the likelihood of convergence to global minimum is increased. The new method can be a good candidate to find the initial condition for the optimization search, i.e., we can use active knots as initial condition of the optimization search to finely tune their optimal location.

References

[1] D. Westwick and R. Kearney, "Separable least squares identification of nonlinear Hammerstein models: Application to stretch reflex dynamics," *Annals of Biomedical Engineering*, vol. 29, pp. 707–718, 2001.

[2] I. W. Hunter and M. J. Korenberg, "The identification of nonlinear biological systems: Wiener and Hammerstein cascade models," *Biological Cybernetics*, vol. 55, no. 2-3, pp. 135–144, 1986.

[3] R. E. Kearney, R. B. Stein, and L. Parameswaran, "Identification of intrinsic and reflex contributions to human ankle stiffness dynamics," *IEEE Transactions on Biomedical Engineering*, vol. 44, no. 6, pp. 493–504, 1997.

[4] G. N. Maksym, R. E. Kearney, and J. H. T. Bates, "Nonparametric block-structured modeling of lung tissue strip mechanics," *Annals of Biomedical Engineering*, vol. 26, pp. 242–252, 1998.

[5] M. Verhaegen and P. Dewilde, "Subspace model identification part 1. the output error state space model identification class of algorithm," *International Journal of control*, vol. 56, no. 5, pp. 1187–1210, 1992.

[6] ——, "Subspace model identification part 2. analysis of the elementary output-error state space model identification algorithm," *International Journal of control*, vol. 56, no. 5, pp. 1211–1241, 1992.

[7] E. Bai and D. Li, "Convergence of the iterative Hammerstein system identification algorithm," *IEEE Transactions on Automatic Control*, vol. 49, no. 11, pp. 1929–1940, 2004.

[8] K. Jalaleddini and R. E. Kearney, "An iterative algorithm for the subspace identification of siso Hammerstein systems," in *Proceedings of IFAC*, 2011, pp. 11 779–11 784.

[9] ——, "Estimation of the gain and threshold of the stretch reflex with a novel subspace identification algorithm," in *Proceedings of IEEE Engineering in Medicine and Biology Society*, 2011, pp. 4431–4434.

[10] R. B. Stein and R. E. Kearney, "Nonlinear behavior of muscle reflexes at the human ankle joint," *Journal of Neurophysiology*, vol. 73, no. 1, pp. 65–72, 1995.

[11] L. Q. Zhang and W. Z. Rymer, "Simultaneous and nonlinear identification of mechanical and reflex properties of human elbow joint muscles," *IEEE Trans Biomed Eng*, vol. 44, no. 12, pp. 1192–1209, 1997.

[12] E. J. Dempsey and D. T. Westwick, "Identification of Hammerstein models with cubic spline nonlinearities," *IEEE Transactions on Biomedical Engineering*, vol. 51, no. 2, pp. 237–245, 2004.

[13] H. Schwetlick and T. Schütze, "Least squares approximation by splines with free knots," *BIT*, vol. 35, no. 3, pp. 361–384, 1995.

[14] L. R. J. Haverkamp, "State space identification: Theory and practice," Ph.D. dissertation, Delft University of Technology, 2001.

[15] W. V. Loock, G. Pipeleers, J. D. Schutter, and J. Swevers, "A convex optimization approach to curve fitting with B-splines," in *Proceedings of IFAC*, 2011, pp. 2290–2295.

[16] M. M. Mirbagheri, H. Barbeau, and R. E. Kearney, "Intrinsic and reflex contributions to human ankle stiffness: variation with activation level and position," *Experimental Brain Research*, vol. 135, no. 4, pp. 423–436, 2000.

[17] I. J. LeonTaritis and S. A. Billings, "Experimental design and identifiability for non-linear systems," *International Journal of System Science*, vol. 18, no. 1, pp. 189–202, 1987.

[18] D. T. Westwick, S. L. Kukreja, and M. J. Brenner, "Identification of highly resonant Hammerstein systems with hard nonlinearities," in *Proceedings of ICNPAA-2006: Mathematical Problems in Engineering and Aerospace Sciences*, 2007, pp. 813–820.