On the use of t-conorms in the gravity-based approach to edge detection

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***Abstract*—This work explores the possibilities of extracting edges using a t-conorm based gravity approach and its relation with the t-norm based one.**

***Keywords*-Edge detection; Gravitational Approach; Dual operators; T-conorms**

I. Introduction

In [1] it was introduced a method for detecting edges adapting the *gravitational approach* by [2]. In that work, the Law of Universal Gravity [3] was modified substituting the product of the masses by the usage of any other t-norm. The method was proven to create different edges depending on the morphological characteristics of each operator. However, the use of the t-norms was only justified by the fact that the product itself is a t-norm.

In this work, we present a study of the t-conorms as substitutes of the t-norms for the *generalized gravitational approach*. The t-norm based edge detector [1] created *weaker* edges in dark zones of the image, while detecting *stronger* ones in the bright ones. Hence, t-norms were more suitable for detecting edges in bright zones of the images. Consequently, high intensity images also tend to have stronger edges than those with low intensity values. In this work we will see how t-conorm based detectors perform in a way diametrically opposed. Besides, we will prove that the information obtained when we use t-norms in the classical composition is the same information that we get when we use t-conorms in the same classical composition.

The use of t-conorms as a replacement for t-norms was a natural choice to explore, as the study of both families [4], [5], [6] is closely related. In concrete, this study is focused on the results obtained with dual operators [7].

The remainder of this paper is organized as follows. Section II includes some previous concepts. In Section III the usage of t-conorms is studied, while Section IV contains some practical results. To finish, some short conclusions are introduced in Section V.

II. Preliminaries

Definition 1: ([5]) A function $f : [0,1]^2 \rightarrow [0,1]$ is called binary aggregation function if it is not decreasing on both arguments and satisfies $f(0,0) = 0$ and $f(1,1) = 1$.

Definition 2: ([5]) A commutative, associative binary aggregation function is called t-norm if it has a neutral element $e = 1$. Alternatively, it is called t-conorm if it has neutral element $e = 0$.

Definition 3: ([5]) An univariate function N defined on $[0,1]$ is called a strong negation if it is strictly decreasing and involutive (i.e.,$N(N(x)) = x$ for all $x \in [0,1]$).

Any strong negation can be represented in terms of automorphism [8] so that $N_\varphi(x) = \varphi^{-1}(1 - \varphi(x))$. For example, $\varphi(x) = x$ generates $N(x) = 1 - x$, while the usage of $\varphi(x) = x^2$ would result in $N(x) = \sqrt{1 - x^2}$.

Definition 4: ([9], [5]) Let $N : [0,1] \rightarrow [0,1]$ be a strong negation and $f : [0,1]^2 \rightarrow [0,1]$ a binary aggregation function. Then, the aggregation function f_d given by

$$f_d(x,y) = N(f(N(x), N(y)))$$

is called the dual of f with respect to N, or, for short, the N-dual of f.

A tuple (f, f_d, N), where f_d is N-dual of f, is a De Morgan triple. Some interesting results are included in [6], while an in-depth study and characterization of these triples can be found in [7].

In this paper we will consider images to be in a L level greyscale, $(0, \ldots, L-1)$. From a given image we could therefore build a fuzzy relation taking as membership the intensity divided by $L-1$. Let Q be an image of $\mathcal{X}$ rows and $\mathcal{Y}$ columns, with $q_{i,j}$ the intensity of the pixel in the position (i,j). We consider $X = \{0, \ldots, \mathcal{X}-1\}$ and $Y = \{0, \ldots, \mathcal{Y}-1\}$ two finite referential sets. Then

$$Q = \{((i,j), \mu_Q(i,j) = q_{i,j}) | (i,j) \in X \times Y\}$$

is called a fuzzy relation on $X \times Y$.

We will represent by $\mathcal{FR}(X \times Y)$ the set of fuzzy relations on the referential set $X \times Y$.

Definition 5: Let N be a strong negation and let $Q \in \mathcal{FR}(X \times Y)$ be a fuzzy relation. The N-*negative* of Q is another fuzzy relation $Q^N \in \mathcal{FR}(X \times Y)$ constructed so that

$$Q^N = \{((i,j), \mu_{Q^N}(i,j) = N(\mu_Q(i,j)) = N(q_{i,j})) | \\ (i,j) \in X \times Y\}$$

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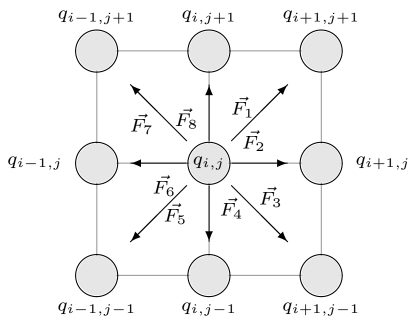

Figure 1. Pixel edge extraction

III. T-CONORM BASED CONSTRUCTIONS

A. The t-norm based approach

In the original *gravitational approach* [2] to edge detection, each pixel in the image is taken as a body of mass equal to its intensity. Then, the position of the pixel is associated a gradient equal to the sum of the gravitational forces its immediate neighbors produce on it. The situation is depicted in Figure 1.

This work will focus on the *feature extraction* stage of an edge detector, as defined by [10]. In this phase, each position of the image gets associated to some representation of the edges. In this case, that information will be a vector, representing both the direction and strength of the intensity change. However, we will not deal with further processing, as non-maximal supression, binarizing or thinning.

In [1] an extension of the original idea was introduced, substituting the product of the masses by any other t-norm in the process of computing the gravitational forces. For example, after Figure 1, force $\vec{F_1}$ is

$$\vec{F_1} = \frac{T(q_{i,j}, q_{i+1,j+1})}{|\vec{r}|^2} \cdot \frac{\vec{r}}{|\vec{r}|} \tag{1}$$

where $\vec{r}$ stands for the vector connecting the pixels (in this case $q_{i,j}$ and $q_{i+1,j+1}$).

The resulting vector associated to position (i, j) would be equal to the sum of all the $\vec{F_k}(i, j)$ with $k \in \{1, .., 8\}$.

$$\vec{F_T}(i, j) = \sum_{k=1}^{8} \vec{F_k} \tag{2}$$

Once the magnitude of the vectors of the image is normalized, each position can be associated 3 different pieces of information. The horizontal and vertical components of the vectors conform two relations $A^x_{T,Q}$ and $A^y_{T,Q}$ in the universal referential set $X \times Y$, so that $A^x_{T,Q}(i, j) \in [-1, 1]$ and $A^y_{T,Q}(i, j) \in [-1, 1]$ for any position $(i, j) \in X \times Y$. On the other hand the magnitude is used as a fuzzy membership in a relation $A_{T,Q} \in \mathcal{FR}(X \times Y)$, with

Figure 2. Edges set and components for Lena image

Figure 3. Original images for edge detection

$$A_{T,Q}(i, j) = \sqrt{(A^x_{T,Q}(i, j))^2 + (A^y_{T,Q}(i, j))^2} \tag{3}$$

Remark 1. Directional components are kept in order to provide extra information about the orientation of the gradient in further steps, as in [11], [12].

Remark 2. Note that only the third of the relations ($A_{T,Q}$) is a fuzzy relation. Directional components relations do contain values in the $[-1, 1]$.

An example of performance is included in Figure 2, where edges in Lena image (256×256 pixels) have been extracted. In Figure 2 both the fuzzy set (left) and the vectorial components (center and right) are included.

However, all of these t-norm based constructions had some problems, as the inability of detecting edges on a 0 intensity pixel, or the different sensitivities when acting on high or low intensity regions of the image. As a result, intensity changes in the bright zones of the image tend to generate higher memberships to the edge set.

To illustrate the problem, we have experimented with Lena images (Figure 3). Two different versions have been derived from them

- Dark version, reduced linearly to the intensities $[0, 0.5]$
- Light version, reduced linearly to the intensities $[0.5, 1]$

Then, generalized gravitational detector has been applied with 3 different t-norms.

- $T_{\mathbf{P}}(x, y) = x \cdot y$
- $T_{\mathbf{M}}(x, y) = \wedge(x, y)$
- $T_{\mathbf{Ł}}(x, y) = \vee(0, x + y - 1)$

Figure 4 contains the results of the experiment. It is evident that edges on brighter versions are usually more evident that those with lower gray intensities, even thought the intensity variations are exactly the same in every position of the image.

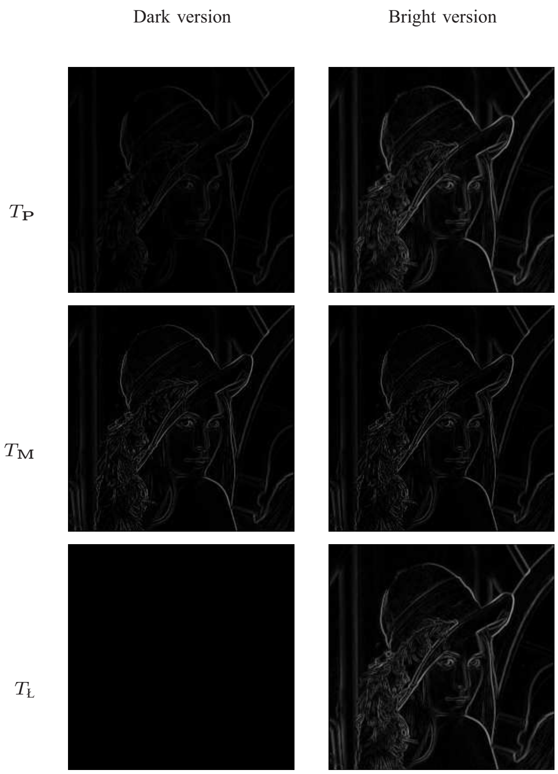

Figure 4. Responses with Lenna image

B. The usage of t-conorms

T-norms and t-conorms are closely related. In fact, a function $T : [0,1]^2 \to [0,1]$ is a t-norm if and only if the function $S : [0,1]^2 \to [0,1]$, defined as $S(x,y) = 1-T(1-x,1-y)$, is a t-conorm [6]. In other words, there exists a bijective application relating the universe of the t-norms and the t-conorms by means of the N-duality, with $N(x) = 1-x$.

When using t-conorms, the key question to be answered is whether there exists some kind of analytical relationship between the edges detected with both families. Furthermore, it is interesting to know whether the use of the t-conorms can detect edges not-detectable using t-norms or not.

In order to make the comparison, we will focus on pairs of N-dual operations, belonging each one to one of the studied families.

Proposition 6: Let (T,S,N) be a De Morgan triple with T a t-norm, S a t-conorm and $N(x) = 1-x$. Under these conditions it is hold

$$\begin{aligned} A^x_{T,Q} + A^x_{S,Q^N} &= 0 \\ A^y_{T,Q} + A^y_{S,Q^N} &= 0 \end{aligned}$$

Proof.

After Figure 1 and equations 1 and 2, we can obtain both of the components of the resulting force in a given position. Considering the construction of the forces, and some trigonometric notions, it is hold that

$$\begin{aligned} A^x_{T,Q}(i,j) = & \underbrace{T(q_{i,j},q_{i+1,j})}_{\alpha 1} - \underbrace{T(q_{i,j},q_{i-1,j})}_{\beta 1} + \\ & \frac{1}{4}\underbrace{T(q_{i,j},q_{i+1,j+1})}_{\alpha 2} - \frac{1}{4}\underbrace{T(q_{i,j},q_{i-1,j-1})}_{\beta 2} + \\ & \frac{1}{4}\underbrace{T(q_{i,j},q_{i+1,j-1})}_{\alpha 3} - \frac{1}{4}\underbrace{T(q_{i,j},q_{i-1,j+1})}_{\beta 3} \end{aligned} \quad (4)$$

where $A^x_{T,Q}(i,j)$ represents the horizontal component of the estimated gradients of the image Q once detected using a t-norm T.

After equation 4, let us apply a t-conorm S instead of T on Q^N.

$$\begin{aligned} A^x_{S,Q^N}(i,j) = & \underbrace{S(N(q_{i,j}),N(q_{i+1,j}))}_{\alpha' 1} - \underbrace{S(N(q_{i,j}),N(q_{i-1,j}))}_{\beta' 1} \\ & + \frac{1}{4}\underbrace{S(N(q_{i,j}),N(q_{i+1,j+1}))}_{\alpha' 2} \\ & - \frac{1}{4}\underbrace{S(N(q_{i,j}),N(q_{i-1,j-1}))}_{\beta' 2} \\ & + \frac{1}{4}\underbrace{S(N(q_{i,j}),N(q_{i+1,j-1}))}_{\alpha' 3} \\ & - \frac{1}{4}\underbrace{S(N(q_{i,j}),N(q_{i-1,j+1}))}_{\beta' 3} \end{aligned} \quad (5)$$

By Definition 4 and considering notation in equations 4 and 5, we know that $\alpha_k = N(\alpha'_k)$ and $\beta_k = N(\beta'_k)$ for any $k \in \{1,2,3\}$.

Then,

$$\begin{aligned} A^x_{T,Q}(i,j) = & \alpha_1 + \frac{1}{4}\alpha_2 + \frac{1}{4}\alpha_3 - \beta_1 - \frac{1}{4}\beta_2 - \frac{1}{4}\beta_3 \\ = & N(\alpha'_1) + \frac{N(\alpha'_2)}{4} + \frac{N(\alpha'_3)}{4} - N(\beta'_1) - \frac{N(\beta'_2)}{4} - \frac{N(\beta'_3)}{4} \\ = & 1 - \alpha'_1 + \frac{1-\alpha'_2}{4} + \frac{1-\alpha'_3}{4} - 1 + \beta'_1 - \frac{1-\beta'_2}{4} - \frac{1-\beta'_3}{4} \\ = & (-1)\left(\alpha'_1 + \frac{1}{4}\alpha'_2 + \frac{1}{4}\alpha'_3 - \beta'_1 - \frac{1}{4}\beta'_2 - \frac{1}{4}\beta'_3\right) \\ = & (-1)A^x_{S,Q^N}(i,j) \end{aligned}$$

This development is equally appliable to the vertical axis.

□

Corollary 7: Under the conditions of Proposition 6, it is hold that

$$A_{T,Q} \quad = \quad A_{S,Q^N}$$

Proof.

It is straightforward combining result of Proposition 6 with formula 3.

□

Therefore, we have the answer for both of the questions we had about t-conorm usage. First at all, can set a relationship between edges extracted with a t-norm and its dual t-conorm. Moreover, it is proven that the edges obtained on an image with a t-norm are exactly the same as those with its dual t-conorm on the negative, with negator $N(x) = 1-x$.

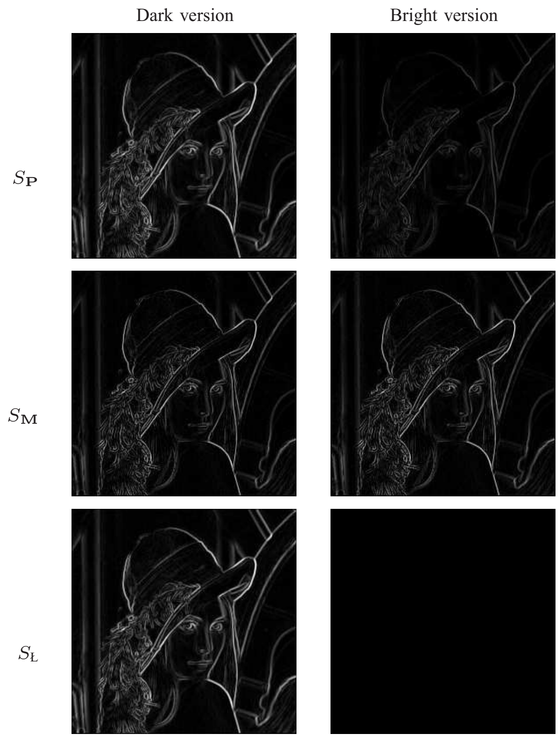

Figure 5. Responses with Lenna images for t-conorms

In practical terms, it means that any information extracted with a t-conorm using the generalized gravitational method can also be extracted by its N dual operator, with $N(x) = 1 - x$. That is, no extra feature information can be extracted by means of t-conorms, other that the one we obtained with t-norms.

This result also offers us an idea about the different performance of the operators in bright and dark regions. After Proposition 6, we know that the sensivity of a t-norm T is reproduced by an equal sensivity of its N-dual t-conorm S in the opposite intensity. That is, in case T is sensitive in the bright regions, so will S in the dark ones.

IV. Results comparison

First at all, we wanted to prove whether the conjunctive operators perform in a way opposite to the t-norms in the bright and dark zones. That is, if the t-conorms perform better in the dark zones of the image than in the bright ones.

We have reproduced experiment with Lena image in bright and dark versions, as in Section III. The results, included in Figure 5, show that the thesis was correct. T-conorms are much more sensitive in dark regions.

An example is here included to illustrate the fact explained in Section III. The pairs of functions used are

- $T_{\mathbf{P}}(x, y) = x \cdot y$ and $S_{\mathbf{P}}(x, y) = x + y - x \cdot y$
- $T_{\mathbf{M}}(x, y) = \wedge(x, y)$ and $S_{\mathbf{M}}(x, y) = \vee(x, y)$

	$T_{\mathbf{P}}$,Q	$T_{\mathbf{P}}$,Q^N	$S_{\mathbf{P}}$,Q	$S_{\mathbf{P}}$,Q^N
$T_{\mathbf{P}}$,Q	1.00	0.93	0.93	1.00
$T_{\mathbf{P}}$,Q^N	0.93	1.00	1.00	0.93
$S_{\mathbf{P}}$,Q	0.93	1.00	1.00	0.93
$S_{\mathbf{P}}$,Q^N	1.00	0.93	0.93	1.00

Table I
Results for T_P and S_P

	$T_Ł$,Q	$T_Ł$,Q^N	$S_Ł$,Q	$S_Ł$,Q^N
$T_Ł$,Q	1.00	0.87	0.87	1.00
$T_Ł$,Q^N	0.87	1.00	1.00	0.87
$S_Ł$,Q	0.87	1.00	1.00	0.87
$S_Ł$,Q^N	1.00	0.87	0.87	1.00

Table II
Results for $T_Ł$ and $S_Ł$

- $T_Ł(x, y) = \vee(0, x + y - 1)$ and $S_Ł(x, y) = \wedge(1, x + y)$

Results are shown in Figure 6. It can be seen how edges in dark (bright) regions edges are better detected with t-conorms (t-norms), as seen in the building or the left lower corner of the cameraman's coat. That is exactly the expected result when using disjunctive operators, instead of conjunctive ones. Nevertheless, results obtained with a t-norm are exactly the same as those taken with its N-dual function on the negative image.

It is also noteworthy that t-norms point out the edges in the brighter object, while the t-conorms do the opposite. Therefore, depending on which family of operators we use the edges of the bright objects are placed either *internally* (in the bright zone) or *externally* (in the dark background). This is a logical consequence of the brightness dependence, which applications are to be studied in the future.

In order not to rely on visual considerations, edges images were compared with the measures introduced in [13], [14] for image comparison. In concrete, we experimented with

$$\mathcal{S}_1(A, B) = \left| \frac{1}{X \times Y} \right| \sum_{(i,j) \in X \times Y} 1 - |(\mu_A(i, j) - \mu_B(i, j))|$$

where $A, B \in \mathcal{FR}(X \times Y)$.

Note that, using this measurement it is hold that

$$\mathcal{S}_1(A, B) = 1 \Leftrightarrow \mu_A(i, j) = \mu_B(i, j) \text{ for any } (i, j) \in X \times Y$$

The results for these measurements with different N-dual pairs of operators are included in tables I, II and III. It is proven that edges fuzzy relations obtained with N-dual operators on N-negative images are exactly the same.

V. Conclusions

We have proven that, when using the gravitatory approach, results obtained with any t-conorm can be reproduced by the means of a t-norms. Therefore, the substitution of the t-norms by t-conorms will not offer new possibilities in edge

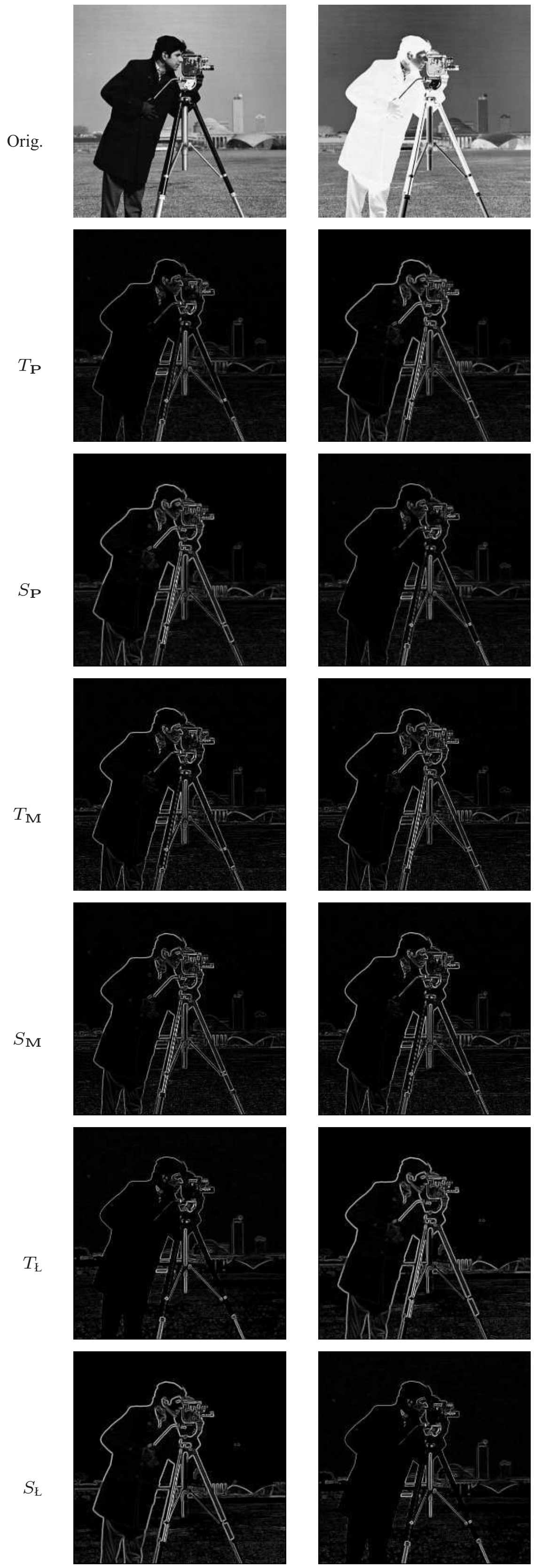

Figure 6. Results for cameraman images with different operators

	$T_{\mathbf{M}}$,Q	$T_{\mathbf{M}}$,Q^N	$S_{\mathbf{M}}$,Q	$S_{\mathbf{M}}$,Q^N
$T_{\mathbf{M}}$,Q	1.00	0.89	0.89	1.00
$T_{\mathbf{M}}$,Q^N	0.89	1.00	1.00	0.89
$S_{\mathbf{M}}$,Q	0.89	1.00	1.00	0.89
$S_{\mathbf{M}}$,Q^N	1.00	0.89	0.89	1.00

Table III
RESULTS FOR T_M AND S_M

detection. This results can be applied not only in the case of t-norms and t-conorms, but also in any further development with N-dual operators, with N a strict negation $N(x) = 1 - x$.

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