Trajectory Tracking of Complex Dynamical Network for Recurrent Neural Network Via Control V-Stability

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***Abstract*— In this paper the problem of trajectory tracking is studied. Based on the V-stability and Lyapunov theory, a control law that achieves the global asymptotic stability of the tracking error between a recurrent neural network and the state of each single node of a complex dynamical network is obtained. To illustrate the analytic results we present a tracking simulation of a simple network with four different nodes and five non-uniform links.**

I. Introduction

The analysis and control of complex behaviors in complex networks, which consist of dynamical nodes, have become a focal point of great interest in recent studies.[1],[2],[3]. The complexity in networks come from structures and dynamics but also their topology often affects their function.

Recurrent neural networks have been widely used in the fields of optimization, pattern recognition, signal processing and control systems, among others. The trajectory tracking is a very interesting problem in the field of theory of system control; it allows us the implementation of important tasks for automatic control such as: high speed target recognition and tracking, real-time visual inspection, and recognition of context sensitive and moving scenes, among others. We present the results of the design of a control law that guarantees the tracking of general complex dynamical networks.

II. Mathematical models

A. General complex dynamical network

Consider a network consisting of N linearly and diffusively coupled nodes, with each node being an n-dimensional dynamical system, described by

$$\dot{x}_i = f_i(x_i) + \sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i), \quad i = 1,2,\ldots,N \quad (1)$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^\top \in \mathbb{R}^n$ are the state vectors of node i, $f_i : \mathbb{R}^n \longmapsto \mathbb{R}^n$ represents the self-dynamics of node i, constants $c_{ij} > 0$ are the coupling strengths between node i and node j, with $i, j = 1,2,\ldots,N$. $\Gamma = (\tau_{ij}) \in \mathbb{R}^{n\times n}$ is a constant internal matrix that describes the way of linking the components in each pair of connected node vectors $(x_j - x_i)$: that is to say for some pairs (i, j) with $1 \le i, j \le n$ and $\tau_{ij} \neq 0$ the two coupled nodes are linked through their ith and jth sub-state variables, respectively, while the coupling matrix $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ denotes the coupling configuration of the entire network: that is to say if there is a connection between node i and node $j (i \neq j)$, then $a_{ij} = a_{ji} = 1$; otherwise $a_{ij} = a_{ji} = 1$.

B. Recurrent neural network

Consider a recurrent neural network in the following form:

$$\begin{aligned} \dot{x}_{i_n} &= A_{i_n} x_{i_n} + W_{i_n} \sigma(x_{i_n}) + u_{i_n} \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^{N} c_{i_n j_n} a_{i_n j_n} \Gamma(x_{j_n} - x_{i_n}), \quad i = 1,2,\ldots,N \end{aligned} \quad (2)$$

where $x_{i_n} = (x_{i_n 1}, x_{i_n 2}, \ldots, x_{i_n n})^\top \in \mathbb{R}^n$ is the state vector of neural network i, $u_{i_n} \in \mathbb{R}^n$ is the input of neural network i, $A_{i_n} = -\lambda_{i_n} I_{n\times n}$, $i = 1,2,\ldots,N$, is the state feedback matrix, with λ_{i_n} being a positive constant, $W_{i_n} \in \mathbb{R}^{n\times n}$ is the connection weight matrix with $i = 1,2,\ldots,N$, and $\sigma(\cdot) \in \mathbb{R}^n$ is a Lipschitz sigmoid vector function [4],[5], such that $\sigma(x_{i_n}) = 0$ only at $x_{i_n} = 0$, with Lipschitz constant L_{σ_i}, $i = 1,2,\ldots,N$ and neuron activation functions $\sigma_i(\cdot) = \tanh(\cdot)$, $i = 1,2,\ldots,n$.

III. Trajectory tracking

The objetive is to develop a control law such that the ith neural network (2) tracks to the trajectory of the ith dynamical system (1). We define the tracking error as $e_i = x_{i_n} - x_i$, $i = 1,2,\ldots,N$ whose time derivative is

$$\dot{e}_i = \dot{x}_{i_n} - \dot{x}_i, \quad i = 1,2,\ldots,N \quad (3)$$

Substituting (1) and (2) in (3), we obtain

$$\begin{aligned} \dot{e}_i &= A_{i_n} x_{i_n} + W_{i_n} \sigma(x_{i_n}) + u_{i_n} - f_i(x_i) \\ &\quad + \sum_{\substack{j=1\\ j\neq i}}^{N} c_{i_n j_n} a_{i_n j_n} \Gamma(x_{j_n} - x_{i_n}) \\ &\quad - \sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i), \quad i = 1,2,\ldots,N \end{aligned} \quad (4)$$

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Adding and substrating $W_{i_n}\sigma(x_i)$, $\alpha_i(t)$, $i=1,2,\ldots,N$, to (4), where α_i to be determined below, and taking into account that $x_{i_n}=e_i+x_i$, $i=1,2,\ldots,N$, then

$$\begin{aligned}\dot{e}_i &= A_{i_n}e_i+W_{i_n}\Big(\sigma(e_i+x_i)-\sigma(x_i)\Big)+(u_{i_n}-\alpha_i)\\&\quad+\Big(A_{i_n}x_i+W_{i_n}\sigma(x_i)+\alpha_i\Big)-f_i(x_i)\\&\quad+\sum_{\substack{j=1\\j\neq i}}^{N}c_{i_nj_n}a_{i_nj_n}\Gamma(x_{j_n}-x_{i_n})\\&\quad-\sum_{\substack{j=1\\j\neq i}}^{N}c_{ij}a_{ij}\Gamma(x_j-x_i),\quad i=1,2,\ldots,N\end{aligned}\tag{5}$$

In order to guarantee that the ith neural network (2) tracks the ith reference trajectory (1), the following assumption has to be satisfed:

Assumption 1. There exist functions $\rho_i(t)$ and $\alpha_i(t)$, $i=1,2,\ldots,N$, such that

$$\begin{aligned}\tfrac{d\rho_i(t)}{dt} &= A_{i_n}\rho_i(t)+W_{i_n}\sigma\big(\rho_i(t)\big)+\alpha_i(t)\\ \rho_i(t) &= x_i(t),\quad i=1,2,\ldots,N\end{aligned}\tag{6}$$

Let define

$$\begin{aligned}\widetilde{u}_{i_n} &= u_{i_n}-\alpha_i\\ \phi_\sigma(e_i,x_i) &= \sigma(e_i+x_i)-\sigma(x_i),\quad i=1,2,\ldots,N\end{aligned}\tag{7}$$

Considering (6) and (7), the equation (5) is reduced to

$$\begin{aligned}\dot{e}_i &= A_{i_n}e_i+W_{i_n}\phi_\sigma(e_i,x_i)+\widetilde{u}_{i_n}\\&\quad+\sum_{\substack{j=1\\j\neq i}}^{N}c_{i_nj_n}a_{i_nj_n}\Gamma(x_{j_n}-x_{i_n})\\&\quad-\sum_{\substack{j=1\\j\neq i}}^{N}c_{ij}a_{ij}\Gamma(x_j-x_i),\quad i=1,2,\ldots,N\end{aligned}\tag{8}$$

Rewriting the summations as

$$\begin{aligned}&\sum_{\substack{j=1\\j\neq i}}^{N}c_{i_nj_n}a_{i_nj_n}\Gamma(x_{j_n}-x_{i_n})\\&=\Gamma\Bigg(\sum_{\substack{j=1\\j\neq i}}^{N}c_{i_nj_n}a_{i_nj_n}x_{j_n}-x_{i_n}\sum_{\substack{j=1\\j\neq i}}^{N}c_{i_nj_n}a_{i_nj_n}\Bigg)\\&\sum_{\substack{j=1\\j\neq i}}^{N}c_{ij}a_{ij}\Gamma(x_j-x_i)\\&=\Gamma\Bigg(\sum_{\substack{j=1\\j\neq i}}^{N}c_{ij}a_{ij}x_j-x_i\sum_{\substack{j=1\\j\neq i}}^{N}c_{ij}a_{ij}\Bigg),\quad i=1,2,\ldots,N\end{aligned}\tag{9}$$

also taking into account that $c_{i_nj_n}=c_{ij}$ and $a_{i_nj_n}=a_{ij}$, then, with the above (8) becomes

$$\begin{aligned}\dot{e}_i &= A_{i_n}e_i+W_{i_n}\phi_\sigma(e_i,x_i)+\widetilde{u}_{i_n}\\&\quad+\Gamma\Bigg(\sum_{\substack{j=1\\j\neq i}}^{N}c_{ij}a_{ij}e_j-e_i\sum_{\substack{j=1\\j\neq i}}^{N}c_{ij}a_{ij}\Bigg)\\&=A_{i_n}e_i+W_{i_n}\phi_\sigma(e_i,x_i)+\widetilde{u}_{i_n}\\&\quad+\sum_{\substack{j=1\\j\neq i}}^{N}c_{ij}a_{ij}\Gamma(e_j-e_i),\quad i=1,2,\ldots,N\end{aligned}\tag{10}$$

It is clear that $e_i=0$, $i=1,2,\ldots,N$ is an equilibrium point of (10), when $\widetilde{u}_{i_n}=0$, $i=1,2,\ldots,N$. To this end, the tracking problem can be restated as a global asymptotic stabilization problem for system (10)

IV. Tracking error stabilization and control design

In order to establish the convergence of (10) to $e_i=0$, $i=1,2,\ldots,N$, which ensures the desired tracking, first, we propose the following candidate Lyapunov function

$$\begin{aligned}V_N(e) &= \sum_{i=1}^{N}V(e_i)=\sum_{i=1}^{N}\tfrac{1}{2}\big\|e_i\big\|^2\\&=\tfrac{1}{2}\sum_{i=1}^{N}e_i^\top e_i,\quad e=\big(e_1^\top,\ldots,e_N^\top\big)^\top\end{aligned}\tag{11}$$

The time derivative of (11), along the trajectories of (10), is

$$\begin{aligned}&\dot{V}_N(e)=\frac{\partial V_N(e)}{\partial e}\dot{e}=\sum_{i=1}^{N}\frac{\partial V_N(e)}{\partial e_i}\dot{e}_i=\big(e_1^\top,\ldots,e_N^\top\big)\times\\&\begin{pmatrix}A_{1_n}e_1+W_{1_n}\phi_\sigma(e_1,x_1)+\widetilde{u}_{1_n}+\sum\limits_{\substack{j=1\\j\neq 1}}^{N}c_{1j}a_{1j}\Gamma(e_j-e_1)\\ \vdots\\ A_{N_n}e_N+W_{N_n}\phi_\sigma(e_N,x_N)+\widetilde{u}_{N_n}+\sum\limits_{\substack{j=1\\j\neq N}}^{N}c_{Nj}a_{Nj}\Gamma(e_j-e_N)\end{pmatrix}\\&=\sum_{i=1}^{N}e_i^\top\Bigg(A_{i_n}e_i+W_{i_n}\phi_\sigma(e_i,x_i)+\widetilde{u}_{i_n}+\sum_{\substack{j=1\\j\neq i}}^{N}c_{ij}a_{ij}\Gamma(e_j-e_i)\Bigg)\end{aligned}\tag{12}$$

Reformulating (12) as

$$\begin{aligned}\dot{V}_N(e) &= \sum_{i=1}^{N}\Big(-\lambda_{i_n}\big\|e_i\big\|^2+e_i^\top W_{i_n}\phi_\sigma(e_i,x_i)+e_i^\top\widetilde{u}_{i_n}\Big)\\&\quad+\sum_{i=1}^{N}\Bigg(\sum_{\substack{j=1\\j\neq i}}^{N}c_{ij}a_{ij}e_i^\top\Gamma e_j-\sum_{\substack{j=1\\j\neq i}}^{N}c_{ij}a_{ij}e_i^\top\Gamma e_i\Bigg)\end{aligned}\tag{13}$$

Next, let consider the following inequality, proved in [6],[7]

$$X^\top Y+Y^\top X\leq X^\top\Lambda X+Y^\top\Lambda^{-1}Y\tag{14}$$

which holds for all matrices $X, Y \in \mathbb{R}^{n\times k}$ and $\Lambda \in \mathbb{R}^{n\times n}$ with $\Lambda = \Lambda^\top > 0$. Applying (14) with $\Lambda = I_{n\times n}$ to the term $e_i{}^\top W_{i_n} \phi_\sigma(e_i, x_i)$, $i = 1, 2, \ldots, N$ we get

$$\begin{aligned} e_i^\top W_{i_n}\phi_\sigma(e_i,x_i) &\leq \tfrac{1}{2}e_i^\top e_i + \tfrac{1}{2}\phi_\sigma^\top(e_i,x_i)W_{i_n}^\top W_{i_n}\phi_\sigma(e_i,x_i) \\ &= \tfrac{1}{2}\|e_i\|^2 e_i + \tfrac{1}{2}\phi_\sigma^\top(e_i,x_i) \\ &\quad \times W_{i_n}^\top W_{i_n}\phi_\sigma(e_i,x_i), \quad i = 1,2,\ldots,N \end{aligned} \tag{15}$$

Taking into account that ϕ_σ is Lypchitz, then

$$\begin{aligned} \|\phi_\sigma(e_i,x_i)\| &= \|\sigma(e_i+x_i) - \sigma(x_i)\| \\ &\leq L_{\phi_{\sigma_i}}\|e_i + x_i - x_i\| \\ &= L_{\phi_{\sigma_i}}\|e_i\|, \quad i = 1,2,\ldots,N \end{aligned} \tag{16}$$

with Lipschitz constant $L_{\phi_{\sigma_i}}$ [4], Applying (16) to $\frac{1}{2}\phi_\sigma^\top(e_i,x_i)W_{i_n}^\top W_{i_n}\phi_\sigma(e_i,x_i)$ we obtain

$$\begin{aligned} &\tfrac{1}{2}\phi_\sigma^\top(e_i,x_i)W_{i_n}^\top W_{i_n}\phi_\sigma(e_i,x_i) \\ \leq\ &\tfrac{1}{2}\|\phi_\sigma^\top(e_i,x_i)W_{i_n}^\top W_{i_n}\phi_\sigma(e_i,x_i)\| \\ \leq\ &\tfrac{1}{2}\left(L_{\phi_{\sigma_i}}\right)^2\|W_{i_n}\|^2\|e_i\|^2, \quad i = 1,2,\ldots,N \end{aligned} \tag{17}$$

Next (15) is reduced to

$$\begin{aligned} &e_i^\top W_{i_n}\phi_\sigma(e_i,x_i) \\ \leq\ &\tfrac{1}{2}\|e_i\|^2 + \tfrac{1}{2}\left(L_{\phi_{\sigma_i}}\right)^2\|W_{i_n}\|^2\|e_i\|^2 \\ =\ &\tfrac{1}{2}\left(1 + L^2_{\phi_{\sigma_i}}\|W_{i_n}\|^2\right)\|e_i\|^2, \quad i = 1,2,\ldots,N \end{aligned} \tag{18}$$

Then we have that

$$\begin{aligned} \dot{V}_N(e) \leq\ & -\sum_{i=1}^{N}\Bigg(\lambda_{i_n}\|e_i\|^2 + \sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}a_{ij}e_i^\top\Gamma e_i\Bigg) \\ & + \tfrac{1}{2}\sum_{i=1}^{N}\Bigg(\left(1 + L^2_{\phi_{\sigma_i}}\|W_{i_n}\|^2\right)\|e_i\|^2 \\ & + 2\sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}a_{ij}e_i^\top\Gamma e_j\Bigg) + \sum_{i=1}^{N} e_i^\top\widetilde{u}_{i_n} \end{aligned} \tag{19}$$

We define $\widetilde{u}_{i_n} = \widetilde{\widetilde{u}}_i + \widetilde{\widetilde{u}}_{i_j}$, $i = 1, 2, \ldots, N$, then (19) becomes

$$\begin{aligned} \dot{V}_N(e) \leq\ & -\sum_{i=1}^{N}\Bigg(\lambda_{i_n}\|e_i\|^2 + \sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}a_{ij}e_i^\top\Gamma e_i\Bigg) \\ & + \tfrac{1}{2}\sum_{i=1}^{N}\left(e_i^\top\left((1 + L^2_{\phi_{\sigma_i}}\|W_{i_n}\|^2)e_i + 2\widetilde{\widetilde{u}}_i\right)\right) \\ & + \sum_{i=1}^{N}\Bigg(e_i^\top\Bigg(\sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}a_{ij}\Gamma e_j + \widetilde{\widetilde{u}}_{i_j}\Bigg)\Bigg) \end{aligned} \tag{20}$$

Now, we propose to use the following control law:

$$\begin{aligned} \widetilde{u}_{i_n} =\ & \widetilde{\widetilde{u}}_i + \widetilde{\widetilde{u}}_{i_j} \\ & -\tfrac{1}{2}\left(1 + L^2_{\phi_{\sigma_i}}\|W_{i_n}\|^2\right)e_i \\ & -\sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}a_{ij}\Gamma e_j, \quad i = 1,2,\ldots,N \end{aligned} \tag{21}$$

then $\dot{V}_N(e) < 0$ for all $e \neq 0$. This means that the proposed control law (21) can globally and asymptotically stabilize the ith error system (10), thereby ensuring the tracking of (1) by (2).

Finally, the control action driven the recurrent neural networks is given by

$$\begin{aligned} u_{i_n} =\ & \widetilde{u}_{i_n} + \alpha_i \\ =\ & -\tfrac{1}{2}\left(1 + L^2_{\phi_{\sigma_i}}\|W_{i_n}\|^2\right)e_i \\ & -\sum_{\substack{j=1\\ j\neq i}}^{N} c_{ij}a_{ij}\Gamma e_j + f_i(x_i) + \lambda_{i_n}x_i \\ & -W_{i_n}\sigma(x_i), \quad i = 1,2,\ldots,N \end{aligned} \tag{22}$$

V. Simulations

In order to illustrate the applicability of the discussed results, we consider a simple network with four different nodes and five non-uniform links. The node self-dynamics are described by [8],[9].

$$\dot{x}_1 = x_1^2, \quad \dot{x}_2 = -3x_2, \quad \dot{x}_3 = \sin x_3, \quad \dot{x}_4 = -|x_4|, \tag{23}$$

and the coupling strengths are $c_{12} = c_{21} = 1.3$, $c_{14} = c_{41} = 1.0$, $c_{13} = c_{31} = 2.7$, $c_{24} = c_{42} = 2.1$, $c_{34} = c_{43} = 1.5$. Fig. 1 shows the divergent phenomenon of network (23) with initial state $X(0) = (0, 0, 10, 0)^\top$ and a three-time stronger coupling strength. The neural network was selected as

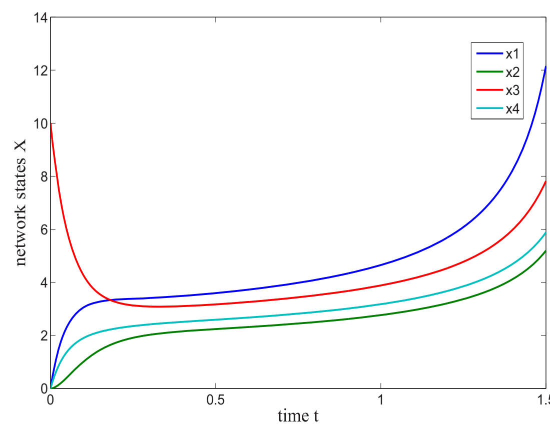

Fig. 1. The evolution of network states with initial state $X(0) = (0, 0, 10, 0)^\top$

$$\begin{aligned} & A_{i_n} = -I_{1\times 1}, \quad W_{i_n} = (1)_{1\times 1}, \quad \sigma(\cdot) = (\tanh x_{i_n})_{1\times 1} \\ & L_{\phi_{\sigma_i}} \triangleq n_i = 1, \quad i = 1,2,3,4 \end{aligned} \tag{24}$$

with initial state $X_n(0) = (0, 0, -10, 0)^\top$ and $\Gamma = I_{1\times 1}$.

The simulation was as follows: for the first 0.5 seconds, the two systems evolute by themselves; in this moment the control law (22) is applied. The results are displayed in the Fig. 2, Fig. 3, Fig. 4 and Fig. 5, and they shown the time evolution for network states respectively. As can be seen the desired tracking is obtained.

VI. Conclusions

We have presented the controller design for trajectory tracking determined by general complex dynamical network. This framework is based on the dynamic neural networks and the methodology is based on V-stability and the Lyapunov theory. The proposed control is applied to a simple network with four different nodes and five non-uniform links, being able to manage to stabilize in a asymptotic form the tracking error between two systems. The results of the simulation show clearly the desired tracking. In a future work it will be considered to be the stochastic case for the complex dynamical network.

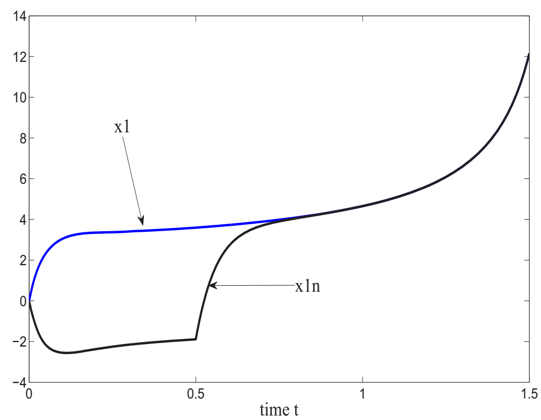

Fig. 2. Time evolution for state 1

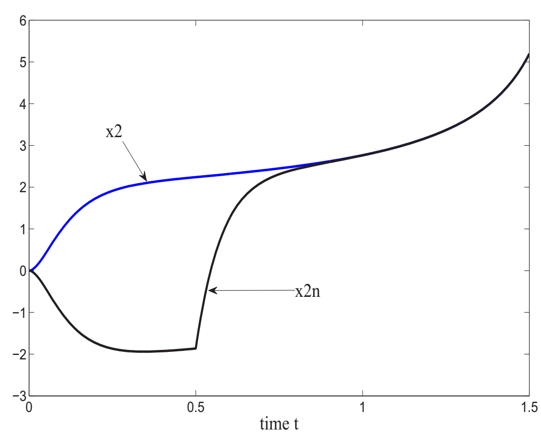

Fig. 3. Time evolution for state 2

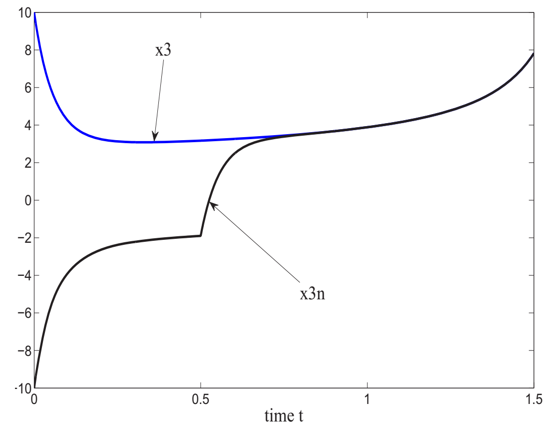

Fig. 4. Time evolution for state 3

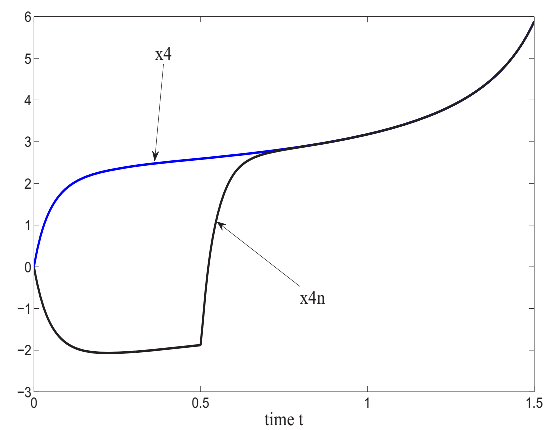

Fig. 5. Time evolution for state 4

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