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Connection among some characterizations of complete fuzzy preorders

Susana Díaz   Davide Martinetti   Ignacio Montes   Susana Montes
Dept. of Statistics and O. R.
Faculty of Science
Calvo Sotelo s/n 33007, Oviedo, Spain
Email: {diazsusana,martinettidavide.uo,uo182551,montes}@uniovi.es

***Abstract*—The concept of (classical) complete preorder can be characterized in several ways. In previous works we have studied whether complete fuzzy preorders can be characterized by the same properties as in the crisp case. We have proven that this is not usually the case. We have studied five possible characterizations and we have proven that only one still characterizes a fuzzy preorder. In this work we study those properties for additive fuzzy preference structures without incomparability. Despite they do not characterize complete fuzzy preorders, they can be related among them. In this contribution we show their connection when the preference structure does not admit incomparable alternatives.**

***Keywords*-Preorder; transitivity; negative transitivity; fuzzy preference relation; completeness condition.**

I. Introduction

Complete preorders are one of the most desirable structures for a set of alternatives to satisfy. When the preferences of a decision maker over a set verify this structure, it is easy to establish an order.

A complete preorder is a complete large preference relation. Transitivity is one of the most important properties in preference modelling. Every crisp reflexive relation can be decomposed into three (two, if the relation is complete) parts: the symmetric, asymmetric and dual symmetric components. The transitivity of a complete reflexive relation is characterized by the transitivity of its symmetric and asymmetric components. Although this is the best known characterization for a complete preorder, there exist other ones.

In [8] five different characterizations of a crisp complete preorder are shown. In [4] we translated those properties to the fuzzy sets context and we studied their connection to a fuzzy strongly complete preorder. Although several general results were presented, we paid special attention to the minimum t-norm. This was the operator considered for generalizing the crisp properties to fuzzy relations. In this contribution we continue the work began there. We consider the five characterizations of a complete preorder defined for fuzzy relations by the two most important t-norms: the minimum and the Łukasiewicz t-norm. For the minimum t-norm, we present some new results that complete the work presented in [4]. For the Łukasiewicz t-norm we carry out a complete study of the connection among the five characterizations considered. The work is developed for additive fuzzy preference structures without incomparability. As we will later recall, this is a way of translating to the fuzzy sets theory the notion of completeness.

The paper is structured in seven sections. Section II contains the characterizations known for complete preorders. In Section III the properties shown in Section II are generalized to fuzzy relations. In Sections IV we recall some results we have proven in previous works and other contributions found in the literature. In Sections V and VI we present the results obtained concerning those properties defined by the minimum and Łukasiewicz t-norms respectively. Section VII closes the work with some conclusions and open points.

II. Characterizations for a crisp preorder

A *large preference relation* is just a reflexive relation R interpreted as follows: aRb *if and only if* a *is preferred or indifferent to* b.

Let us denote the transpose, the complement and the dual of a relation R by R^t, R^c and R^d, respectively. Crisp reflexive relations allow to build three disjoint relations: the strict preference relation $P = R \cap R^d$, the indifference relation $I = R \cap R^t$ and the incomparability relation $J = R^c \cap R^d$. The triplet (P, I, J) forms a *preference structure*. From this triplet the original reflexive relation can be obtained: $R = P \cup I$. Thus, every preference structure has associated a unique large preference relation that completely characterizes the preference structure. A crisp relation Q defined on A is complete if $A^2 = Q \cup Q^t$. The completeness of R is equivalent to the absence of incomparability ($J = \emptyset$) in the associated preference structure.

The composition of two relations Q_1 and Q_2 is the binary relation $Q_1 \circ Q_2$ defined by $Q_1 \circ Q_2(a, c) = \sup_b(\min(Q_1(a, b), Q_2(b, c)))$. The transitivity of Q is equivalent to $Q \circ Q \subseteq Q$.

The relation Q is called negatively transitive if it holds that $aQc \Rightarrow (aQb \vee bQc)$ for all a, b, c.

A binary relation Q defined on a set A, can be represented by the graph (A, Q) where A is the set of nodes and Q the set of arcs, i.e. there is an arc from the node a to the node b if and only if aQb and it is represented as (a, b). A path of length

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n in such a graph is a set of n arcs $(a_0, a_1), \ldots, (a_{n-1}, a_n)$ in (A, Q). A *circuit* in (A, Q) is a path for which $a_0 = a_n$. Given a complete binary relation R on A and the associated preference structure (P, I), the following statements are equivalent [8]:

1) P and I are transitive,
2) P is transitive and $P \circ I \subseteq P$,
3) P is transitive and $I \circ P \subseteq P$,
4) P is negatively transitive,
5) there is no P in circuits of length ≤ 3 in (A, R),
6) R is transitive.

Let us note that, since P is irreflexive, there is no P in circuits of length 1. On the other hand, if for two elements a and b it holds that aRb and bRa then aIb and there is no P in the circuit $(a, b), (b, a)$. Then, Property 5 can be written as *every circuit of length* 3 *in* (A, R) *contains no* P.

A complete reflexive relation R is a *preorder* if it satisfies the transitive property. The first five conditions above provide five characterizations of a preorder.

III. Fuzzy relations

A. Fuzzy preference structures

In fuzzy set theory, a reflexive fuzzy relation R on A can also be decomposed into the so-called (additive) fuzzy preference structure, by means of a generator i. This was defined by De Baets and Fodor in [1] as a symmetric (commutative) mapping $i : [0,1]^2 \to [0,1]$ bounded by the Łukasiewicz t-norm, $T_{\mathbf{L}}$, and the minimum operator, $T_{\mathbf{M}}$, i.e. $T_{\mathbf{L}} \leq i \leq T_{\mathbf{M}}$.

Given a reflexive fuzzy relation R and a generator i, the three components of an additive fuzzy preference structure (AFPS) are defined as follows:

$$\begin{aligned}
P(a,b) &= p(R(a,b), R(b,a)) \\
&= R(a,b) - i(R(a,b), R(b,a))\,, \\
I(a,b) &= i(R(a,b), R(b,a))\,, \\
J(a,b) &= j(R(a,b), R(b,a)) \\
&= I(a,b) - (R(a,b) + R(b,a) - 1)\,.
\end{aligned}$$

They satisfy the additive property: $P(a,b) + I(a,b) + P^t(a,b) + J(a,b) = 1$ for all $a, b \in A$. The corresponding large preference relation R from which they are defined is then given by $R(a,b) = P(a,b) + I(a,b)$.

The concept of completeness for fuzzy relations is usually defined by a t-conorm. The most usual completeness conditions considered are the *strong completeness*, defined by the maximum t-conorm: Q is strongly complete if $\max(Q(a,b), Q(b,a)) = 1$ for all $a, b \in A$; and the *weak completeness* defined by the Łukasiewicz t-conorm: Q is weakly complete if $Q(a,b) + Q(b,a) \geq 1$ for all $a, b \in A$. The absence of the associated incomparability relation is not equivalent to any completeness condition over the reflexive fuzzy relation R.

Lemma 3.1: [6] Let R be a reflexive fuzzy relation and let J be the incomparability relation associated to R by means of any generator i. Then the following equivalence holds

$$J = \emptyset \Leftrightarrow \begin{cases} R \text{ is weakly complete} \\ i|_S = T_{\mathbf{L}} \end{cases}$$

where $S = \{(u,v) \in [0,1]^2 : \exists (x,y) \in A^2 \text{ with } R(x,y) = u, R(y,x) = v\}$.

Strong completeness is a more restrictive condition than the absence of incomparability relation and this is a stronger property than weak completeness. As we explained at the beginning in this work we focus on additive fuzzy preference structures without incomparability. Therefore, we handle weakly complete reflexive relations R such that the associated additive fuzzy preference structure $(P, I, \emptyset)$ is defined by the Łukasiewicz generator:

$$(P, I) = (R^d, R \cap_{T_{\mathrm{L}}} R^t).$$

B. Fuzzyfication of properties

The composition of fuzzy relations is usually defined by t-norms. Any t-norm T leads to a definition. The T-composition of two fuzzy relations Q_1 and Q_2 on A is defined by $Q_1 \circ_T Q_2\,(a,c) = \sup_b T(Q_1(a,b), Q_2(b,c))$. The definition can be extended to any conjunctor (see [4], [5]) but in this work we will restrict to t-norms. The definition of transitivity also depends on the t-norm (in general, conjunctor) we choose. Given the t-norm T, Q is *T-transitive* if

$$T(Q(a,b), Q(b,c)) \leq Q(a,c)\,, \quad \forall\, a, b, c.$$

As for crisp relations, the T-transitivity of a fuzzy relation is equivalent to $Q \circ_T Q \subseteq Q$.

Let us recall the definition of negative S-transitivity (see for example [7]). Given a t-conorm S, the fuzzy relation Q is negatively S-transitive if $Q(a,c) \leq S(Q(a,b), Q(b,c))$ for all a, b, c.

Concerning the absence of strict preference in cycles of length 3, this property states that

$$aRb \wedge bRc \wedge cRa \quad \Rightarrow \quad a \not{P} b \wedge b \not{P} c \wedge c \not{P} a, \quad \forall a, b, c.$$

As commented above, the intersection of two fuzzy relations depends on the t-norm considered. Then Condition 5 of Section II depends on the t-norm T fixed. Given a reflexive fuzzy relation R on A and a t-norm T we say that *no circuit of length 3 in* (A, R) *contains* P if it holds that

$$\begin{aligned}
&T(1 - P(a,b), 1 - P(b,c), 1 - P(c,a)) \\
&\qquad \geq T(R(a,b), R(b,c), R(c,a))\,.
\end{aligned}$$

If we recall that the dual t-conorm S of a t-norm T is defined as $S(x,y) = 1 - T(1-x, 1-y)$, then the previous expression can also be written as

$$\begin{aligned}
&1 - S(P(a,b), P(b,c), P(c,a)) \\
&\qquad \geq T(R(a,b), R(b,c), R(c,a))\,,
\end{aligned}$$

We have identified Property 5 of Section II for fuzzy relations with the absence of strict preference in cycles of length 3 in (A, R) . The absence of strict preferences in cycles of length 1 and 2 is always guaranteed. Since P is asymmetric, it cannot be involved in cycles of length 1. Concerning cycles of length 2, by the additive property, $R(a,b) \leq 1 - P(b,a)$ for all a, b, and this implies $T(1-P(a,b), 1-P(b,a)) \geq T(R(b,a), R(a,b) = T(R(a,b), R(b,a))$.

Thus, given a t-norm T and its dual t-conorm S, the five characterizations presented in Section II can be written for fuzzy relations as follows:

(I) $\begin{cases} P \text{ is } T\text{-transitive,} \\ I \text{ is } T\text{-transitive;} \end{cases}$

(II) $\begin{cases} P \text{ is } T\text{-transitive,} \\ P \circ_T I \subseteq P; \end{cases}$

(III) $\begin{cases} P \text{ is } T\text{-transitive,} \\ I \circ_T P \subseteq P; \end{cases}$

(IV) P is negatively T-transitive: for all a, b, c
$P(a,c) \leq S(P(a,b), P(b,c))$

(V) every circuit of length 3 in (A, R) contains no P: for all a, b, c
$1 - S(P(a,b), P(b,c), P(c,a))$
$\geq T(R(a,b), R(b,c), R(c,a))$

IV. KNOWN RESULTS

Next we recall some results that are already known.

When the large preference relation is strongly complete, some good results concerning the transitivity of R can be obtained. For example, the following equivalence holds.

Proposition 4.1: [2] Let R be a strongly complete reflexive relation, let $(P, I, \emptyset)$ be its associated AFPS and $T \geq T_{\mathrm{L}}$ a t-norm. Then,

$$R \circ_T R \subseteq R \quad \Leftrightarrow \quad \begin{cases} P \circ_{T_{\mathrm{M}}} P \subseteq P\,, \\ I \circ_T I \subseteq I\,, \\ P \circ_{T_{\mathrm{L}}} I \subseteq P\,, \\ I \circ_{T_{\mathrm{L}}} P \subseteq P\,. \end{cases}$$

However, most of the good properties that hold in the strongly complete case fail when dealing with a more general range of AFPS. In this section we recall the results known for AFPS without incomparability. Let us recall that these are the AFPS generated from a weakly complete reflexive relation by the Łukasiewicz t-norm.

Concerning the transitivity of P we have already proven in [5] that the $T_{\mathbf{M}}$-transitivity of R does not imply the $T_{\mathbf{M}}$-transitivity of P. In fact, for weakly complete reflexive relations and when the generator is the Łukasiewicz t-norm, the greatest t-norm that can be obtained from the $T_{\mathbf{M}}$-transitivity of R is the $T_{\mathbf{nM}}$-transitivity of P, where $T_{\mathbf{nM}}$ is the nilpotent minimum t-norm. As a direct consequence, the $T_{\mathbf{M}}$-transitivity of R does not guarantee any one of the conditions (I), (II), (III).

Concerning the converse implication we had proven the following result.

Proposition 4.2: [4] Let R be a reflexive fuzzy relation and P and I the corresponding strict preference and indifference relations obtained from R by means of a generator i. It holds that

$$\left.\begin{array}{l} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ I \circ_{T_{\mathrm{M}}} I \subseteq I \\ P \circ_{T_{\mathrm{M}}} I \subseteq P \\ I \circ_{T_{\mathrm{M}}} P \subseteq P \end{array}\right\} \Rightarrow R \circ_{T_{\mathrm{M}}} R \subseteq R\,.$$

It is also known the equivalence between the transitivity of the large preference relation and the negative transitivity of its strict preference relation. The following equivalence can be found in [7], [9].

Proposition 4.3: Let R be a fuzzy relation and T a t-norm. It holds that

$$R \text{ is } T\text{-transitive } \Leftrightarrow R^d \text{ is negatively } T\text{-transitive.}$$

This equivalence can be generalized to any commutative conjunctor as we showed in [4]. When the additive fuzzy preference structure does not admit incomparable alternatives, it holds that $P = R^d$. So we can state the following direct corollary.

Corollary 4.4: [4] Let R be a fuzzy relation and T a t-norm. It holds that

$$R \text{ is } T\text{-transitive } \Leftrightarrow P \text{ is negatively } T\text{-transitive.}$$

Next results concern property (V).

Proposition 4.5: [4] Let R be a weakly complete reflexive fuzzy relation and P the corresponding strict preference relation generated from R by means of $i = T_{\mathrm{L}}$. It holds that

$$R \text{ is } T_{\mathrm{M}}\text{-transitive}$$

$$\Downarrow$$

$$1 - S_{\mathrm{M}}(P(a,b), P(b,c), P(c,a)) \geq$$
$$T_{\mathrm{M}}(R(a,b), R(b,c), R(c,a)), \quad \forall\, a, b, c.$$

The converse implication was also studied. We proved in [4] that it does not hold for any conjunctor f. It does not hold in particular for any t-norm.

Proposition 4.6: Let R be a weakly complete reflexive fuzzy relation and $(P, I, \emptyset)$ its associated AFPS. Let T be a t-norm. Then,

$$1 - T^d(P(a,b), P(b,c), P(c,a)) \geq$$
$$T(R(a,b), R(b,c), R(c,a)), \quad \forall\, a, b, c$$

$$\not\Downarrow$$

$$R \text{ is } T\text{-transitive.}$$

Let us recall that the implication holds for any t-norm for the particular case of strongly complete reflexive fuzzy relations.

In Figure 1 we summarize the known characterizations for T_{M}-transitive large preference relations.

$$\left.\begin{array}{l} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ I \circ_{T_{\mathrm{M}}} I \subseteq I \\ P \circ_{T_{\mathrm{M}}} I \subseteq P \\ I \circ_{T_{\mathrm{M}}} P \subseteq P \end{array}\right\}$$

$$\Downarrow$$

$$P \text{ es neg. } T_{\mathrm{M}}\text{-transitive} \Leftrightarrow R \circ_{T_{\mathrm{M}}} R \subseteq R$$

$$\Downarrow$$

$$1 - S_{\mathrm{M}}(P(a,b), P(b,c), P(c,a)) \geq$$
$$T_{\mathrm{M}}(R(a,b), R(b,c), R(c,a)), \quad \forall\, a,b,c.$$

Figure 1. Connection among the T_{M}-transitivity of the large preference relation associated to a fuzzy preference structure without incomparability and Properties (I)-(V).

V. THE MINIMUM T-NORM

The results presented in the previous section can be improved. In particular, given a preference structure without incomparability, it holds that the four conditions imposed in Proposition 4.2 are redundant.

Proposition 5.1: Let $(P, I, \emptyset)$ be an additive fuzzy preference structure without incomparability. Then,

$$\left.\begin{array}{l} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ P \circ_{T_{\mathrm{M}}} I \subseteq P \end{array}\right\} \Rightarrow \left\{\begin{array}{l} I \circ_{T_{\mathrm{M}}} P \subseteq P \\ I \circ_{T_{\mathrm{M}}} I \subseteq I \end{array}\right.$$

It also holds that

$$\left.\begin{array}{l} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ I \circ_{T_{\mathrm{M}}} P \subseteq P \end{array}\right\} \Rightarrow \left\{\begin{array}{l} P \circ_{T_{\mathrm{M}}} I \subseteq P \\ I \circ_{T_{\mathrm{M}}} I \subseteq I \end{array}\right.$$

We have also proven that there exist additive fuzzy preference structures such that P and I are T_{M}-transitive but $P \circ_{T_{\mathrm{M}}} I \not\subseteq P$ and $I \circ_{T_{\mathrm{M}}} P \not\subseteq P$.

Therefore,

Corollary 5.2: Let $(P, I, \emptyset)$ be an additive fuzzy preference structure without incomparability. Then,

$$\left.\begin{array}{l} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ I \circ_{T_{\mathrm{M}}} P \subseteq P \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ P \circ_{T_{\mathrm{M}}} I \subseteq P \end{array}\right.$$
$$\Downarrow$$
$$\left\{\begin{array}{l} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ I \circ_{T_{\mathrm{M}}} I \subseteq I \end{array}\right.$$

The implication missing (from above to up) does not hold.

It also follows from Propositions 4.2 and 5.1 that

Corollary 5.3: Let R be a reflexive fuzzy relation and P and I the corresponding strict preference and indifference relations generated from R by means of a generator i. It holds that

$$\left.\begin{array}{l} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ P \circ_{T_{\mathrm{M}}} I \subseteq P \end{array}\right\} \Rightarrow R \circ_{T_{\mathrm{M}}} R \subseteq R\,.$$

It is clear that the implication still holds if we replace $P \circ_{T_{\mathrm{M}}} I \subseteq P$ by $I \circ_{T_{\mathrm{M}}} P \subseteq P$.

$$\left.\begin{array}{l} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ P \circ_{T_{\mathrm{M}}} I \subseteq P \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ I \circ_{T_{\mathrm{M}}} P \subseteq P \end{array}\right\}$$

$$\Swarrow \qquad \Searrow$$

$$P \text{ is neg. } T_{\mathrm{M}}\text{-transitive} \qquad \left\{\begin{array}{l} P \circ_{T_{\mathrm{M}}} P \subseteq P \\ I \circ_{T_{\mathrm{M}}} I \subseteq I \end{array}\right.$$

$$\Downarrow$$

$$1 - S_{\mathrm{M}}(P(a,b), P(b,c), P(c,a)) \geq$$
$$T_{\mathrm{M}}(R(a,b), R(b,c), R(c,a)), \quad \forall\, a,b,c.$$

Figure 2. Relationships among Properties (I)-(V) for the minimum t-norm. The implications missing do not hold.

VI. THE ŁUKASIEWICZ T-NORM

In this section we will focus on the Łukasiewicz t-norm. This is one of the most important t-norms, since it has shown very good properties. Let us recall for example Proposition 4.1 or Lemma 3.1. Next we use this t-norm to define all the possible characterizations of a complete preorder shown in Section III. We study their relationships. As we will see later, the results are quite different from the ones presented in the previous section.

We begin by the connection between the negative transitivity of the strict preference relation and the transitivity of the strict preference and indifference relations.

Proposition 6.1: [5] Let R be a weakly complete reflexive relation and $(P, I, \emptyset)$ its associated by $i = T_{\mathrm{L}}$ additive fuzzy preference structure. Then,

$$R \circ_{T_{\mathrm{L}}} R \subseteq R \Rightarrow \left\{\begin{array}{l} P \circ_{T_{\mathrm{L}}} P \subseteq P \\ I \circ_{T_{\mathrm{L}}} I \subseteq I \end{array}\right.$$

As a consequence of this result and Corollary 4.4 we get the following result.

Corollary 6.2: Let $(P, I, \emptyset)$ be an additive fuzzy preference structure without incomparability. Then

$$P \text{ negatively } T_{\mathrm{L}}\text{-transitive } \Rightarrow \left\{\begin{array}{l} P \circ_{T_{\mathrm{L}}} P \subseteq P \\ I \circ_{T_{\mathrm{L}}} I \subseteq I \end{array}\right.$$

In [3] it was proven the following:

Proposition 6.3: Let R be a weakly complete reflexive relation and $(P, I, \emptyset)$ its associated additive fuzzy preference structure. Then,

$$R \circ_{T_{\mathrm{L}}} R \subseteq R \Rightarrow \begin{cases} P \circ_{T_{\mathrm{L}}} I \subseteq P \\ I \circ_{T_{\mathrm{L}}} P \subseteq P \end{cases}$$

This result joined to Corollary 4.4 leads to the following implication.

Corollary 6.4: Let $(P, I, \emptyset)$ be an additive fuzzy preference structure without incomparability. Then

$$P \text{ negatively } T_{\mathrm{L}}\text{-transitive } \Rightarrow \begin{cases} P \circ_{T_{\mathrm{L}}} I \subseteq P \\ I \circ_{T_{\mathrm{L}}} P \subseteq P \end{cases}$$

Therefore, condition (IV) implies conditions (I), (II) and (III).

The converse implications to the ones presented above do not hold. In [3] it was proven that

$$\left.\begin{array}{l} P \circ_{T_{\mathrm{L}}} P \subseteq P \\ I \circ_{T_{\mathrm{L}}} I \subseteq I \\ P \circ_{T_{\mathrm{L}}} I \subseteq P \\ I \circ_{T_{\mathrm{L}}} P \subseteq P \end{array}\right\} \not\Rightarrow R \circ_{T_{\mathrm{L}}} R \subseteq R$$

The counterexample presented, together with Proposition 4.4 allows to state that condition (IV) does not follow from properties (I), (II) or (III). Moreover, it does not follow from conditions (I), (II) and (III) together.

Properties (IV) and (V) are not related. We can provide counterexamples for both implications.

Property (V) does guarantee that the strict preference relation is T_{L}-transitive.

Proposition 6.5: Let $(P, I, \emptyset)$ be an additive fuzzy preference structure without incomparability and R its associated large preference relation. It holds that

$$1 - S_{\mathrm{L}}(P(a,b), P(b,c), P(c,a)) \geq$$
$$T_{\mathrm{L}}(R(a,b), R(b,c), R(c,a)), \quad \forall\, a, b, c.$$
$$\Downarrow$$
$$P \circ_{T_{\mathrm{L}}} P \subseteq P$$

However,

$$1 - S_{\mathrm{L}}(P(a,b), P(b,c), P(c,a)) \geq$$
$$T_{\mathrm{L}}(R(a,b), R(b,c), R(c,a)), \quad \forall\, a, b, c.$$
$$\not\Downarrow$$
$$\begin{cases} I \circ_{T_{\mathrm{L}}} I \subseteq I \\ P \circ_{T_{\mathrm{L}}} I \subseteq P \\ I \circ_{T_{\mathrm{L}}} P \subseteq P \end{cases}$$

Therefore, property (V) does not imply (I), (II) or (III).

The converse implication neither holds. Moreover,

$$\left.\begin{array}{l} P \circ_{T_{\mathrm{L}}} P \subseteq P \\ I \circ_{T_{\mathrm{L}}} I \subseteq I \\ I \circ_{T_{\mathrm{L}}} P \subseteq P \\ P \circ_{T_{\mathrm{L}}} I \subseteq P \end{array}\right\} \not\Rightarrow$$
$$1 - S_{\mathrm{L}}(P(a,b), P(b,c), P(c,a)) \geq$$
$$T_{\mathrm{L}}(R(a,b), R(b,c), R(c,a)), \quad \forall\, a, b, c.$$

$$\begin{array}{lcl} & \not\nearrow & \begin{cases} P \circ_{T_{\mathrm{L}}} P \subseteq P \\ I \circ_{T_{\mathrm{L}}} I \subseteq I \end{cases} \\ P \text{ is neg. } T_{\mathrm{L}}\text{-transitive} & \Rightarrow & \begin{cases} P \circ_{T_{\mathrm{L}}} P \subseteq P \\ I \circ_{T_{\mathrm{L}}} P \subseteq P \end{cases} \\ & \not\searrow & \begin{cases} P \circ_{T_{\mathrm{L}}} P \subseteq P \\ P \circ_{T_{\mathrm{L}}} I \subseteq P \end{cases} \end{array}$$

$$1 - S_{\mathrm{L}}(P(a,b), P(b,c), P(c,a)) \geq$$
$$T_{\mathrm{L}}(R(a,b), R(b,c), R(c,a)), \quad \forall\, a, b, c.$$

Figure 3. Relationships among Properties (I)-(V) for the Łukasiewicz t-norm. The implications missing do not hold.

VII. Conclusion

We have considered five known characterizations for crisp complete preorders. We have studied the connection among fuzzy counterparts of those characterizations. We have focused on additive fuzzy preference structures without incomparability. We have used the two most important t-norms for defining those properties for fuzzy relations: the minimum and the Łukasiewicz t-norms. We have checked that the results obtained are quite different. For the Łukasiewicz t-norm the negative transitivity of P seems to be a strong condition, while for the minimum t-norm it is weaker than the transitivity of P joined to the condition the composition of P and I is a subset of P.
The next natural step is to generalize the study to other t-norms.

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