Measuring social welfare through location and consensus measures

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Abstract

In this paper we introduce a new procedure for comparing and ordering social welfare situations by considering location, dispersion, consensus and welfare measures generated by exponential means. These measures satisfy interesting properties and generalize some measures used in welfare economics.

1. Introduction

In the economic literature there exists a large variety of economic indices (inequality, poverty, social development, gross domestic product, happiness, health, welfare, etc.) for comparing and ranking social situations across populations (see, for instance, Sen [13], Dagum [6] and Chakravarty and Muliere [4, 5]).

Along the paper we assume that economic indices have been normalized into the unit interval.

Given two societies with the same aggregated economic index, it seems reasonable to rank first that society with a smaller dispersion. For instance, if a society has two members and each one has half chicken, then the mean is the same that another society of two members where one of them has one chicken and the other one has no chicken. Taking into account the dispersion of both societies, it is clear that the first society should be ranked before the second one regarding equality and social welfare. This simple idea was developed by Sen [12] by multiplying the mean, as location measure, by 1 minus the Gini coefficient, as consensus measure.

In this paper we follow the seminal idea of Sen, but we use the cores of exponential means as location measures and some consensus measures generated by the anti-self-dual remainders associated with the corresponding exponential means (see García-Lapresta and Marques Pereira [8]). Taking into account these two ingredients, both satisfying interesting properties, we define a new social welfare order on societies.

The paper is organized as follows. Section 2 includes basic notation and properties of aggregation functions. Section 3 is devoted to summarize the decomposition of an aggregation function in the core and the associated remainder. In Section 4 we introduce our proposal for measuring social welfare by considering location, dispersion, consensus and welfare measures generated by exponential means. Finally, Section 5 contains some concluding remarks.

2 Aggregation functions

In this section we present notation and basic definitions regarding aggregation functions on $[0,1]^n$, with $n \in \mathbb{N}$ and $n \geq 2$ throughout the text.

Notation Points in $[0,1]^n$ will be denoted by means of boldface characters: $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{0} = (0, \ldots, 0)$. For $x \in [0,1]$, we have $x \cdot \mathbf{1} = (x, \ldots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i \in \{1, \ldots, n\}$; by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Moreover, $x_* = \min\{x_1, \ldots, x_n\}$ and $x^* = \max\{x_1, \ldots, x_n\}$. Given a permutation on $\{1, \ldots, n\}$, i.e., a bijection $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, with $\boldsymbol{x}_\sigma$ we denote $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

We begin by defining standard properties of real functions on $[0,1]^n$. On this, see Fodor and Roubens [7], Calvo *et al.* [3], Beliakov *et al.* [2], García-Lapresta and Marques Pereira [8] and Grabisch *et al.* [9].

Definition 1 *Let* $A : [0,1]^n \longrightarrow \mathbb{R}$ *be a function.*

1. *A is* idempotent *if for every* $x \in [0,1]$*:*

$$A(x \cdot \mathbf{1}) = x.$$

2. *A is* symmetric *if for every permutation* σ *on* $\{1, \ldots, n\}$ *and every* $\boldsymbol{x} \in [0,1]^n$*:*

$$A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x}).$$

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3. A *is* monotonic *if for all* $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$*:*

$$\boldsymbol{x} \geq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \geq A(\boldsymbol{y}).$$

4. A *is* strictly monotonic *if for all* $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$*:*

$$\boldsymbol{x} > \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) > A(\boldsymbol{y}).$$

5. A *is* compensative *if for every* $\boldsymbol{x} \in [0,1]^n$*:*

$$x_* \leq A(\boldsymbol{x}) \leq x^*.$$

6. A *is* self-dual *if for every* $\boldsymbol{x} \in [0,1]^n$*:*

$$A(\mathbf{1} - \boldsymbol{x}) = 1 - A(\boldsymbol{x}).$$

7. A *is* anti-self-dual *if for every* $\boldsymbol{x} \in [0,1]^n$*:*

$$A(\mathbf{1} - \boldsymbol{x}) = A(\boldsymbol{x}).$$

8. A *is* invariant for translations *if for all* $t \in [-1,1]$ *and* $\boldsymbol{x} \in [0,1]^n$*:*

$$A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x})$$

whenever $\boldsymbol{x} + t \cdot \mathbf{1} \in [0,1]^n$.

9. A *is* stable for translations *if for all* $t \in [-1,1]$ *and* $\boldsymbol{x} \in [0,1]^n$*:*

$$A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x}) + t$$

whenever $\boldsymbol{x} + t \cdot \mathbf{1} \in [0,1]^n$.

Let now $\{A^{(k)}\}_{k \in \mathbb{N}}$ *be a sequence of functions, where* $A^{(k)} : [0,1]^k \longrightarrow \mathbb{R}$.

10. $\{A^{(k)}\}_{k \in \mathbb{N}}$ *is* invariant for replications *if for all* $\boldsymbol{x} \in [0,1]^n$ *and any number of replications* $m \in \mathbb{N}$ *of* $\boldsymbol{x}$*:*

$$A^{(mn)}(\overbrace{\boldsymbol{x}, \ldots, \boldsymbol{x}}^{m}) = A^{(n)}(\boldsymbol{x}).$$

Definition 2 *A function* $A : [0,1]^n \longrightarrow [0,1]$ *is called an* n-ary aggregation function *if it is monotonic and satisfies* $A(\mathbf{0}) = 0$ *and* $A(\mathbf{1}) = 1$*. For the sake of simplicity, the* n*-arity is omitted whenever it is clear from the context. An aggregation function is said to be* strict *if it is strictly monotonic.*

Self-duality and stability for translations are important properties of aggregation functions. In turn, anti-self-duality and invariance for translations are incompatible with the defining properties of aggregation functions, namely with the boundary conditions $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$. Nevertheless, anti-self-duality and invariance for translations play an important role in this paper in so far as they are properties of important functions associated with aggregation functions, as we shall discuss later. The following are standard facts concerning aggregation functions.

Proposition 1 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function.*

1. *A is idempotent if and only if A is compensative.*
2. *If A is strict, then* $A(\boldsymbol{x}) = 0$ *if and only if* $\boldsymbol{x} = \mathbf{0}$*, and* $A(\boldsymbol{x}) = 1$ *if and only if* $\boldsymbol{x} = \mathbf{1}$.
3. *If A is stable for translations, then A is idempotent.*

3. The self-dual core and the associated remainder

In this section we briefly summarize the decomposition of an aggregation function in the self-dual core and the associated remainder included in García-Lapresta and Marques Pereira [8]. First we introduce the concepts of self-dual core and anti-self-dual remainder of an aggregation function, establishing which properties are inherited in each case from the original aggregation function. Particular emphasis is given to the properties of stability for translations (self-dual core) and invariance for translations (anti-self-dual remainder).

Definition 3 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function. The aggregation function* $A^* : [0,1]^n \longrightarrow [0,1]$ *defined as*

$$A^*(\boldsymbol{x}) = 1 - A(\mathbf{1} - \boldsymbol{x})$$

is known as the dual *of the aggregation function A.*

Clearly, $(A^*)^* = A$, which means that dualization is an *involution*. An aggregation function A is self-dual if and only if $A^* = A$. The properties of idempotency, symmetry, strict monotonicity, compensativeness, self-duality, anti-self-duality, invariance and stability for translations are all preserved by duality. The same holds for continuity. Aggregation functions are not in general self-dual. However, a self-dual aggregation function can be associated to any aggregation function in a simple manner. The construction of the so-called *self-dual core* of an aggregation function A is as follows.

Definition 4 *Let* $A : [0,1]^n \longrightarrow [0,1]$ *be an aggregation function. The function* $\widehat{A} : [0,1]^n \longrightarrow [0,1]$ *defined by*

$$\widehat{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) + A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) - A(\mathbf{1} - \boldsymbol{x}) + 1}{2}.$$

is called the core *of the aggregation function A.*

Notice that $\widehat{A}$ is clearly an aggregation function, verifying the boundary conditions $\widehat{A}(\mathbf{0}) = 0$, $\widehat{A}(\mathbf{1}) = 1$ and monotonicity. Moreover, $\widehat{A}$ is self-dual, since $\widehat{A}(\mathbf{1} - \boldsymbol{x}) = 1 - \widehat{A}(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$. We say that $\widehat{A}$ is the *self-dual core* of the aggregation function A.

Proposition 2 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function.*

1. *A is self-dual if and only if $\widehat{A}(\boldsymbol{x}) = A(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$.*

2. *Accordingly, $\widehat{\widehat{A}}(\boldsymbol{x}) = \widehat{A}(\boldsymbol{x})$ for every $\boldsymbol{x} \in [0,1]^n$.*

Proposition 3 *The self-dual core $\widehat{A}$ inherits from the aggregation function A the properties of idempotency (hence, compensativeness), symmetry, strict monotonicity, continuity, stability for translations and invariance for replications, whenever A has these properties.*

We now introduce the *anti-self-dual remainder* $\widetilde{A}$, which is simply the difference between the original aggregation function A and its self-dual core $\widehat{A}$.

Definition 5 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. The function $\widetilde{A} : [0,1]^n \longrightarrow \mathbb{R}$ defined by $\widetilde{A}(\boldsymbol{x}) = A(\boldsymbol{x}) - \widehat{A}(\boldsymbol{x})$, that is*

$$\widetilde{A}(\boldsymbol{x}) = \frac{A(\boldsymbol{x}) - A^*(\boldsymbol{x})}{2} = \frac{A(\boldsymbol{x}) + A(\mathbf{1} - \boldsymbol{x}) - 1}{2},$$

is called the remainder *of the aggregation function A.*

Notice that $\widetilde{A}$ is anti-self-dual. For this reason we say that $\widetilde{A}$ is the *anti-self-dual remainder* of the aggregation function A. Clearly, $\widetilde{A}$ is not an aggregation function. In particular, $\widetilde{A}(\mathbf{0}) = \widetilde{A}(\mathbf{1}) = 0$ violates the boundary conditions and implies that $\widetilde{A}$ is either non monotonic or everywhere null. Moreover, $-0.5 \le \widetilde{A}(\boldsymbol{x}) \le 0.5$ for every $\boldsymbol{x} \in [0,1]^n$.

Proposition 4 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function.*

1. *A is self-dual if and only if $\widetilde{A}(\boldsymbol{x}) = 0$ for every $\boldsymbol{x} \in [0,1]^n$.*

2. *Accordingly, $\widetilde{\widehat{A}}(\boldsymbol{x}) = 0$ for every $\boldsymbol{x} \in [0,1]^n$.*

Proposition 5 *The remainder $\widetilde{A}$ inherits from the aggregation function A the properties of symmetry, continuity and invariance for replications, whenever A has these properties.*

The remainder $\widetilde{A}$ is symmetric, whenever the aggregation function A has that property. The same holds for continuity. Summarizing, every aggregation function A decomposes additively $A = \widehat{A} + \widetilde{A}$ in two components: the self-dual core $\widehat{A}$ and the anti-self-dual remainder $\widetilde{A}$, where only $\widehat{A}$ is an aggregation function. The so-called *dual decomposition* $A = \widehat{A} + \widetilde{A}$ clearly shows some analogy with other algebraic decompositions, such as that of square matrices and bilinear tensors into their symmetric and skew-symmetric components. The following result concerns two more properties of the anti-self-dual remainder based directly on the definition $\widetilde{A} = A - \widehat{A}$ and the corresponding properties of the self-dual core.

Proposition 6 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function.*

1. *If A is idempotent, then $\widetilde{A}(x \cdot \mathbf{1}) = 0$ for every $x \in [0,1]$.*

2. *If A is stable for translations, then $\widetilde{A}$ is invariant for translations.*

These properties of the anti-self-dual remainder are suggestive. The first statement applies to the class of idempotent aggregation functions. In such case, self-dual cores are idempotent and therefore anti-self-dual remainders are null on the main diagonal. The second statement applies to the subclass of stable aggregation functions. In such case, self-dual cores are stable and therefore anti-self-dual remainders are invariant for translations. In other words, if the aggregation function A is stable for translations, the value $\widetilde{A}(\boldsymbol{x})$ does not depend on the average value of the $\boldsymbol{x}$ coordinates, but only on their numerical deviations from that average value. These properties of the anti-self-dual remainder $\widetilde{A}$ suggest that it may give some indication on the dispersion of the $\boldsymbol{x}$ coordinates.

4. Exponential means

Quasiarithmetic means are aggregation functions that satisfy interesting properties. They were simultaneously characterized in 1930 by Kolmogoroff [10] and Nagumo [11] (see also Fodor and Roubens [7, pp. 112-114]). Exponential means are quasiarithmetic means that are stable for translations. Thus, their cores are also stable for translations, and their anti-self-dual remainders are invariant for translations, joint with other interesting properties. This is the reason why we consider these functions for defining appropriate location and dispersion measures in our setting.

Definition 6 *Let $A : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. We say that A is a* quasiarithmetic mean *if there exists an order automorphism (bijective and increasing function) $\varphi : [0,1] \longrightarrow [0,1]$ such that*

$$A(\boldsymbol{x}) = \varphi^{-1}\left(\frac{\varphi(x_1) + \cdots + \varphi(x_n)}{n}\right)$$

where φ is said to generate the quasiarithmetic mean A.

Exponential means are the quasiarithmetic means generated by the order automorphisms $\varphi_\alpha,\ \alpha \neq 0$

$$\varphi_\alpha(x) = \frac{e^{\alpha x} - 1}{e^{\alpha} - 1}.$$

The limit case, $\varphi_0(x) = x$, generates the arithmetic mean.

Exponential means joint with the arithmetic mean are the only quasiarithmetic means that are stable for translations.

Definition 7 *Given $\alpha \neq 0$, the exponential mean A_α is the aggregation function defined by*

$$A_\alpha(\boldsymbol{x}) = \frac{1}{\alpha} \ln \frac{\sum_{i=1}^{n} e^{\alpha x_i}}{n}.$$

Proposition 7 *For every $\alpha \neq 0$, A_α is idempotent, symmetric, strictly monotonic, compensative, stable for translations and invariant for replications.*

Definition 8 *Given $\alpha \neq 0$, the* location measure *associated with A_α is the self-dual core of A_α*

$$L_\alpha(\boldsymbol{x}) = \widehat{A}_\alpha(\boldsymbol{x}) = \frac{1}{2\alpha} \ln \frac{\sum_{i=1}^{n} e^{\alpha x_i}}{\sum_{i=1}^{n} e^{-\alpha x_i}}.$$

Proposition 8 *For every $\alpha \neq 0$, L_α is idempotent, symmetric, strictly monotonic, compensative, stable for translations, self-dual and invariant for replications.*

Definition 9 *Given $\alpha \neq 0$, the* dispersion measure *associated with A_α is the anti-self-dual remainder of A_α*

$$D_\alpha(\boldsymbol{x}) = \widetilde{A}_\alpha(\boldsymbol{x}) = \frac{1}{2\alpha} \ln \frac{\sum_{i=1}^{n} e^{\alpha x_i} \cdot \sum_{i=1}^{n} e^{-\alpha x_i}}{n^2}.$$

Proposition 9 *For every $\alpha \neq 0$, $D_\alpha(\boldsymbol{x}) = 0$ if and only if $x_1 = \cdots = x_n$. Moreover, D_α is symmetric, anti-self-dual, invariant for translations and invariant for replications.*

Remark 1 If $\alpha > 0$, then $0 \leq D_\alpha(\boldsymbol{x}) \leq 0.5$ for every $\boldsymbol{x} \in [0,1]^n$. If $\alpha < 0$, then $-0.5 \leq D_\alpha(\boldsymbol{x}) \leq 0$ for every $\boldsymbol{x} \in [0,1]^n$.

Definition 10 *Given $\alpha \neq 0$, the* consensus measure *associated with A_α is the function $C_\alpha : [0,1]^n \longrightarrow [0,1]$ defined by*

$$C_\alpha(\boldsymbol{x}) = 1 - 2\,|D_\alpha(\boldsymbol{x})|.$$

Proposition 10 *For every $\alpha \neq 0$, $C_\alpha(\boldsymbol{x}) = 1$ if and only if $x_1 = \cdots = x_n$. Moreover, C_α is symmetric, anti-self-dual, invariant for translations and invariant for replications.*

The following result presents the limits of the exponential means and the associated location, dispersion and consensus measures. The proof is by straightforward application of l'Hospital's rule.

Proposition 11 *The following statements hold:*

1. $\lim_{\alpha \to \infty} A_\alpha(\boldsymbol{x}) = x^*$.
2. $\lim_{\alpha \to -\infty} A_\alpha(\boldsymbol{x}) = x_*$.
3. $\lim_{\alpha \to 0} A_\alpha(\boldsymbol{x}) = \dfrac{x_1 + \cdots + x_n}{n}$.
4. $\lim_{\alpha \to \infty} L_\alpha(\boldsymbol{x}) = \lim_{\alpha \to -\infty} L_\alpha(\boldsymbol{x}) = \dfrac{x_* + x^*}{2}$.
5. $\lim_{\alpha \to 0} L_\alpha(\boldsymbol{x}) = \dfrac{x_1 + \cdots + x_n}{n}$.
6. $\lim_{\alpha \to \infty} D_\alpha(\boldsymbol{x}) = \dfrac{x^* - x_*}{2}$.
7. $\lim_{\alpha \to -\infty} D_\alpha(\boldsymbol{x}) = -\dfrac{x^* - x_*}{2}$.
8. $\lim_{\alpha \to 0} D_\alpha(\boldsymbol{x}) = 0$.
9. $\lim_{\alpha \to \infty} C_\alpha(\boldsymbol{x}) = \lim_{\alpha \to -\infty} C_\alpha(\boldsymbol{x}) = 1 - (x^* - x_*)$.
10. $\lim_{\alpha \to 0} C_\alpha(\boldsymbol{x}) = 1$.

We now introduce the welfare measure associated with an aggregation function as the location measure corrected by the consensus measure (a factor which diminishes as inequality increases).

Definition 11 *Given $\alpha \neq 0$, the* welfare measure *associated with A_α is the function $W_\alpha : [0,1]^n \longrightarrow [0,1]$ defined by*

$$W_\alpha(\boldsymbol{x}) = L_\alpha(\boldsymbol{x}) \cdot C_\alpha(\boldsymbol{x}).$$

Proposition 12 *For every $\alpha \neq 0$, W_α is idempotent, symmetric and invariant for replications.*

Taking into account the previous measures, it is possible to compare populations of different size. In the following definition, we order populations by means of the welfare measure. So, populations with the same location measure may be ordered by considering their dispersion. Even more, a population with a smaller location measure than a second one may be ordered before the last one if the consensus is sufficiently bigger than the second one. This idea is already in Sen [12] by using the arithmetic mean and 1 minus the Gini coefficient as location and consensus measures, respectively.

Definition 12 *Given* $\boldsymbol{x}, \boldsymbol{y} \in \cup_{n\in\mathbb{N}}[0,1]^n$,

$$\boldsymbol{x} \succsim_\alpha \boldsymbol{y} \ \Leftrightarrow\ W_\alpha(\boldsymbol{x}) \geq W_\alpha(\boldsymbol{y}).$$

Obviously, $\succsim_\alpha$ is a weak order.

Example 1 Consider 9 populations whose economic indices are included in the first column of Tables 1-5. Notice that the first six populations share the same average, $\frac{1}{3}$, and the averages of the other ones are close to this amount. The last three columns of these Tables show the location, consensus and welfare measures for $\alpha = \frac{1}{4}, \frac{1}{2}, 1, 2, 4$. Table 6 includes the rankings produced by the corresponding weak orders $\succsim_\alpha$. Notice that these rankings have some differences. This is due to the importance that parameter α gives to dispersion.

Table 1. Values for $\alpha = 0.25$

$\alpha = 0.25$	$L_\alpha(\boldsymbol{x})$	$C_\alpha(\boldsymbol{x})$	$W_\alpha(\boldsymbol{x})$
$\boldsymbol{x}_1 = (1,0,0)$	0.3341	0.9445	0.3155
$\boldsymbol{x}_2 = (\frac{1}{2},\frac{1}{2},0)$	0.3332	0.9861	0.3286
$\boldsymbol{x}_3 = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$	0.3333	1	0.3333
$\boldsymbol{x}_4 = (1,\frac{1}{3},0,0)$	0.3339	0.9583	0.3200
$\boldsymbol{x}_5 = (\frac{2}{3},\frac{2}{3},0,0)$	0.3333	0.9722	0.3241
$\boldsymbol{x}_6 = (\frac{4}{9},\frac{4}{9},\frac{4}{9},0)$	0.3332	0.9907	0.3301
$\boldsymbol{x}_7 = (0.87,0.23,0.13,0.1)$	0.3328	0.9753	0.3246
$\boldsymbol{x}_8 = (0.7,0.15,0.14,0.11)$	0.2751	0.9848	0.2710
$\boldsymbol{x}_9 = (0.56,0.5,0.15,0.1)$	0.3275	0.9895	0.3240

Table 2. Values for $\alpha = 0.5$

$\alpha = 0.5$	$L_\alpha(\boldsymbol{x})$	$C_\alpha(\boldsymbol{x})$	$W_\alpha(\boldsymbol{x})$
$\boldsymbol{x}_1 = (1,0,0)$	0.3364	0.8897	0.2992
$\boldsymbol{x}_2 = (\frac{1}{2},\frac{1}{2},0)$	0.3329	0.9723	0.3237
$\boldsymbol{x}_3 = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$	0.3333	1	0.3333
$\boldsymbol{x}_4 = (1,\frac{1}{3},0,0)$	0.3356	0.9170	0.3077
$\boldsymbol{x}_5 = (\frac{2}{3},\frac{2}{3},0,0)$	0.3333	0.9447	0.3149
$\boldsymbol{x}_6 = (\frac{4}{9},\frac{4}{9},\frac{4}{9},0)$	0.3330	0.9815	0.3268
$\boldsymbol{x}_7 = (0.87,0.23,0.13,0.1)$	0.3339	0.9508	0.3174
$\boldsymbol{x}_8 = (0.7,0.15,0.14,0.11)$	0.2757	0.9698	0.2674
$\boldsymbol{x}_9 = (0.56,0.5,0.15,0.1)$	0.3275	0.9792	0.3207

Table 3. Values for $\alpha = 1$

$\alpha = 1$	$L_\alpha(\boldsymbol{x})$	$C_\alpha(\boldsymbol{x})$	$W_\alpha(\boldsymbol{x})$
$\boldsymbol{x}_1 = (1,0,0)$	0.3447	0.7838	0.2702
$\boldsymbol{x}_2 = (\frac{1}{2},\frac{1}{2},0)$	0.3318	0.9448	0.3135
$\boldsymbol{x}_3 = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$	0.3333	1	0.3333
$\boldsymbol{x}_4 = (1,\frac{1}{3},0,0)$	0.3421	0.8357	0.2859
$\boldsymbol{x}_5 = (\frac{2}{3},\frac{2}{3},0,0)$	0.3333	0.8909	0.2970
$\boldsymbol{x}_6 = (\frac{4}{9},\frac{4}{9},\frac{4}{9},0)$	0.3320	0.9630	0.3197
$\boldsymbol{x}_7 = (0.87,0.23,0.13,0.1)$	0.3379	0.9020	0.3048
$\boldsymbol{x}_8 = (0.7,0.15,0.14,0.11)$	0.2778	0.9398	0.2610
$\boldsymbol{x}_9 = (0.56,0.5,0.15,0.1)$	0.3275	0.9585	0.3139

Table 4. Values for $\alpha = 2$

$\alpha = 2$	$L_\alpha(\boldsymbol{x})$	$C_\alpha(\boldsymbol{x})$	$W_\alpha(\boldsymbol{x})$
$\boldsymbol{x}_1 = (1,0,0)$	0.3702	0.5995	0.2220
$\boldsymbol{x}_2 = (\frac{1}{2},\frac{1}{2},0)$	0.3276	0.8919	0.2922
$\boldsymbol{x}_3 = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$	0.3333	1	0.3333
$\boldsymbol{x}_4 = (1,\frac{1}{3},0,0)$	0.3635	0.6852	0.2491
$\boldsymbol{x}_5 = (\frac{2}{3},\frac{2}{3},0,0)$	0.3333	0.7925	0.2642
$\boldsymbol{x}_6 = (\frac{4}{9},\frac{4}{9},\frac{4}{9},0)$	0.3281	0.9266	0.3040
$\boldsymbol{x}_7 = (0.87,0.23,0.13,0.1)$	0.3521	0.8081	0.2845
$\boldsymbol{x}_8 = (0.7,0.15,0.14,0.11)$	0.2855	0.8810	0.2515
$\boldsymbol{x}_9 = (0.56,0.5,0.15,0.1)$	0.3276	0.9186	0.3009

5 Concluding remarks

In Economics, there exist a long tradition of aggregate different economic indicators in order to make pairwise comparisons among societies or countries. In this issue, we have considered location, dispersion, consensus and welfare measures associated with exponential means, the only class of quasiarithmetic means satisfying stability for translations. It is worth noting that in our proposal each exponential mean generates a location, a dispersion, a consensus and a welfare measure. Depending on the parameter we use, those measures have different sensitivity towards inequality (as in Atkinson [1]).

Table 5. Values for $\alpha = 4$

$\alpha = 4$	$L_\alpha(\boldsymbol{x})$	$C_\alpha(\boldsymbol{x})$	$W_\alpha(\boldsymbol{x})$
$\boldsymbol{x}_1 = (1, 0, 0)$	0.4167	0.3647	0.1520
$\boldsymbol{x}_2 = (\frac{1}{2}, \frac{1}{2}, 0)$	0.3149	0.7998	0.2518
$\boldsymbol{x}_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	0.3333	1	0.3333
$\boldsymbol{x}_4 = (1, \frac{1}{3}, 0, 0)$	0.4095	0.4617	0.1890
$\boldsymbol{x}_5 = (\frac{2}{3}, \frac{2}{3}, 0, 0)$	0.3333	0.6463	0.2154
$\boldsymbol{x}_6 = (\frac{4}{9}, \frac{4}{9}, \frac{4}{9}, 0)$	0.3151	0.8578	0.2703
$\boldsymbol{x}_7 = (0.87, 0.23, 0.13, 0.1)$	0.3893	0.6510	0.2534
$\boldsymbol{x}_8 = (0.7, 0.15, 0.14, 0.11)$	0.3087	0.7749	0.2392
$\boldsymbol{x}_9 = (0.56, 0.5, 0.15, 0.1)$	0.3277	0.8482	0.2780

Table 6. Rankings

	$\succsim_{0.25}$	$\succsim_{0.5}$	$\succsim_{1}$	$\succsim_{2}$	$\succsim_{4}$
1	x_3	x_3	x_3	x_3	x_3
2	x_6	x_6	x_6	x_6	x_9
3	x_2	x_2	x_9	x_9	x_6
4	x_7	x_9	x_2	x_2	x_7
5	x_5	x_7	x_7	x_7	x_2
6	x_9	x_5	x_5	x_5	x_8
7	x_4	x_4	x_4	x_8	x_5
8	x_1	x_1	x_1	x_4	x_4
9	x_8	x_8	x_8	x_1	x_1

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