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# The Fuzzy Probabilistic Weighted Averaging Operator and its Application in Decision Making

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## Abstract

*We present a new aggregation operator that uses the probability and the weighted average in the same formulation. Moreover, we consider a situation where the information is uncertain and can be represented with fuzzy numbers. We call this new aggregation operator the fuzzy probabilistic weighted average (FPWA) operator. We study some of its main properties. We also study its applicability and we focus on a business decision making problem about the selection of monetary policies.*

## 1. Introduction

The weighted average (WA) is one of the most common aggregation operators found in the literature. It can be used in a wide range of different problems including statistics, economics, engineering, etc. Another aggregation process very common in the literature is the use of probabilistic information in the aggregation. Both models are very useful for solving a wide range of problems. For further reading on these and other aggregation operators, see for example [1-2,5-6,8-12,14-16].

Usually, when using these approaches it is considered that the available information are exact numbers. However, this may not be the real situation found in the specific problem considered. Sometimes, the available information is vague or imprecise and it is not possible to analyze it with exact numbers. Then, it is necessary to use another approach that is able to assess the uncertainty such as the use of fuzzy numbers (FNs). In order to develop the fuzzy approach, we will follow the ideas of [3-4,7-9,13,17-18]. Note that in the literature, there are a lot of studies dealing with uncertain information represented in the form of FNs in different problems such as [4,7-9].

Recently [9], Merigó has suggested a new approach that unifies the probability and the WA in the same formulation. He called it the probabilistic weighted averaging (PWA) operator. This unification permits to include both concepts in the aggregation and considering the degree of importance that they have in the problem. Thus, for the extreme cases, we also find the probability and the WA as particular cases of this approach. Therefore, as it was explained in [9], all the previous studies developed with probabilities or with WAs can be extended with this new formulation.

In this paper, we suggest a new approach of the PWA operator for uncertain situations that cannot be assessed with exact numbers but it is possible to use FNs. We introduce the fuzzy probabilistic weighted average (FPWA) operator. It is an aggregation function that unifies the probability and the WA in the same formulation in an environment where the available information is given in the form of FNs. We study some of the main properties of the FPWA operator and different particular cases.

We also study the applicability of the FPWA operator and it has been explained before for the PWA, it is applicable in a wide range of problems. More specifically, it is applicable in all the studies where it is possible to use probabilities or WAs under an uncertain environment represented with FNs. Note that it generalizes all the situations included in the PWA with exact numbers because the exact number is a particular case of the FN. Moreover, all the situations that can be represented with interval numbers are also included in this formulation.

We develop an application in a decision making problem about political management where a government is looking for the optimal monetary policy for the next period.

This paper is organized as follows. In Section 2, we briefly describe the FNs, the probabilistic aggregation operators and the weighted aggregation operators. Section 3 presents the FPWA operator. Section 4 analyzes some particular cases and Section 5 develops

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an application of the new approach. Finally, we summarize the main conclusions in Section 6.

## 2. Preliminaries

In this Section we briefly review the FNs, the probabilistic and the weighted aggregation operators.

### 2.1. Fuzzy Numbers

A FN $A$ is defined as a fuzzy subset of a universe of discourse that is both convex (i.e., $\mu_A(\lambda x_1 + (1-\lambda)x_2 \geq \min(\mu_A(x_1), \mu_A(x_2))$; for $\forall x_1, x_2 \in R$ and $\lambda \in [0, 1]$) and normal (i.e., $\sup_{x\in R} \mu_A(x) = 1$).

Note that the FN may be considered as a generalization of the interval number although it is not strictly the same because the interval numbers may have different meanings. In the literature, we find a wide range of FNs [4,7-9] such as the Triangular FN (TFN), the Trapezoidal FN (TpFN), the Interval-Valued FN (IVFN), the Generalized FN (GFN), etc.

For example, a TpFN $A$ of a universe of discourse $R$ can be characterized by a trapezoidal membership function ($\alpha$–cut representation) $A = (\underline{a}, \overline{a})$ such that

$$\begin{aligned}\underline{a}(\alpha) &= a_1 + \alpha(a_2 - a_1),\\ \overline{a}(\alpha) &= a_4 - \alpha(a_4 - a_3).\end{aligned} \tag{1}$$

where $\alpha \in [0, 1]$ and parameterized by $(a_1, a_2, a_3, a_4)$ where $a_1 \leq a_2 \leq a_3 \leq a_4$, are real values. Note that if $a_1 = a_2 = a_3 = a_4$, then, the FN is a crisp value and if $a_2 = a_3$, the FN is represented by a TFN. Note that the TFN can be parameterized by $(a_1, a_2, a_4)$.

In the following, we are going to review some basic FN arithmetic operations as follows. Let $A$ and $B$ be two TFNs, where $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$.

1. $A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
2. $A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$
3. $A \times k = (k \times a_1, k \times a_2, k \times a_3)$; for $k > 0$.

Note that other operations could be studied but in this paper we will focus on these ones. For more complete information and overviews about FNs, see for example [4,7,9].

### 2.2. Probabilistic Aggregation Functions

Probabilistic aggregation functions (or operators) are those functions that use probabilistic information in the aggregation process. Some examples are the aggregation with simple probabilities, the aggregation with belief structures [9], the concept of immediate probabilities [5,8] and the probabilistic OWA operator [9,11]. The immediate probability is an approach that uses OWAs and probabilities in the same formulation. It can be defined as follows.

**Definition 1.** An IPOWA operator of dimension $n$ is a mapping IPOWA: $R^n \rightarrow R$ that has an associated weighting vector $W$ of dimension $n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$\text{IPOWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{2}$$

where $b_j$ is the $j$th largest of the $a_i$, each argument $a_i$ has a probability $v_i$ with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = (w_j v_j / \sum_{j=1}^{n} w_j v_j)$ and $v_j$ is the probability $v_i$ ordered according to $b_j$, that is, according to the $j$th largest of the $a_i$.

Note that the IPOWA operator is a good approach for unifying probabilities and OWAs in some particular situations. But it is not always useful, especially in situations where we want to give more importance to the probabilities or to the OWA operators. In order to see why this unification does not seem to be a final model is considering other ways of representing $\hat{v}_j$. For example, we could also use $\hat{v}_j = [w_j + v_j / \sum_{j=1}^{n}(w_j + v_j)]$ or other similar approaches.

Another approach for unifying probabilities and OWAs in the same formulation is the probabilistic OWA (POWA) operator [11]. Its main advantage is that it is able to include both concepts considering the degree of importance of each case in the problem. It is defined as follows.

**Definition 2.** A POWA operator of dimension $n$ is a mapping POWA: $R^n \rightarrow R$ that has an associated weighting vector $W$ of dimension $n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$\text{POWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{p}_j b_j \tag{3}$$

where $b_j$ is the $j$th largest of the $a_i$, each argument $a_i$ has an associated probability $p_i$ with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, $\hat{p}_j = \beta w_j + (1-\beta) p_j$ with $\beta \in [0, 1]$ and $p_j$

is the probability $p_i$ ordered according to the $j$th largest of the $a_i$.

### 2.3. Weighted Aggregation Functions

Weighted aggregation functions are those functions that weight the aggregation process by using the weighted average. Some examples are the aggregation with the weighted average, with belief structures that use the weighted average [9] and with the weighted OWA (WOWA) operator [14]. The weighted average can be defined as follows.

**Definition 3.** A WA operator of dimension $n$ is a mapping WA: $R^n \rightarrow R$ that has an associated weighting vector $W$, with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, such that

$$\text{WA}(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_i \tag{4}$$

where $a_i$ represents the $i$th argument variable.

Other extensions of the weighted average are those that use it with the OWA operator such as the WOWA operator and the hybrid averaging (HA) operator [15]. Recently [10], Merigó suggested another approach called the OWA weighted average (OWAWA) operator. Its main advantage against the WOWA and the HA is that it includes the OWA and the WA considering the degree of importance that each concept have in the aggregation. It can be defined as follows.

**Definition 4.** An OWAWA operator of dimension $n$ is a mapping OWAWA: $R^n \rightarrow R$ that has an associated weighting vector $W$ of dimension $n$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:

$$\text{OWAWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j \tag{5}$$

where $b_j$ is the $j$th largest of the $a_i$, each argument $a_i$ has an associated weight (WA) $v_i$ with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0, 1]$ and $v_j$ is the weight (WA) $v_i$ ordered according to $b_j$, that is, according to the $j$th largest of the $a_i$.

Note that other approaches for unifying the OWA and the WA are possible as it was suggested in [9] such as a similar approach than the immediate probability. Thus, in the WA we get the immediate weighted OWA (IWOWA) operator that could be defined, for example, by using $\hat{v}_j = (w_j v_j / \sum_{j=1}^{n} w_j v_j)$ or by using $\hat{v}_j = [w_j + v_j / \sum_{j=1}^{n} (w_j + v_j)]$.

Note that in the literature we find a lot of extensions of weighted aggregation functions such as those that use uncertain information represented in the form of interval numbers, FNs or linguistic variables [2,9].

## 3. The Fuzzy Probabilistic Weighted Averaging Operator

The fuzzy probabilistic weighted averaging (FPWA) operator is an aggregation operator that unifies the probability and the weighted average in the same formulation considering the degree of importance that each concept has in the aggregation process. Moreover, it is also able to deal with uncertain environments that can be assessed with different types of FNs. It is defined as follows.

**Definition 5.** Let $\Psi$ be the set of FNs. A FPWA operator of dimension $n$ is a mapping FPWA: $\Psi^n \rightarrow \Psi$ such that:

$$\text{FPWA}(\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} \hat{v}_i \tilde{a}_i \tag{6}$$

where the $\tilde{a}_i$ are the argument variables represented in the form of FNs, each argument $\tilde{a}_i$ has an associated weight (FWA) $v_i$ with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, and a probabilistic weight $p_i$ with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$, $\hat{v}_i = \beta p_i + (1-\beta) v_i$ with $\beta \in [0, 1]$ and $\hat{v}_i$ is the weight that unifies probabilities and WAs in the same formulation.

Note that it is also possible to formulate the FPWA operator separating the part that strictly affects the probabilistic information and the part that affects the FWAs. This representation is useful to see both models in the same formulation but it does not seem to be as a unique equation that unifies both models.

**Definition 6.** Let $\Psi$ be the set of FNs. A FPWA operator is a mapping FPWA: $\Psi^n \rightarrow \Psi$ of dimension $n$, if it has an associated probabilistic vector $P$, with $\sum_{i=1}^{n} p_i = 1$ and $p_i \in [0, 1]$ and a weighting vector $V$ that affects the FWA, with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, such that:

$$f(\tilde{a}_1, \ldots, \tilde{a}_n) = \beta \sum_{j=1}^{n} p_i \tilde{a}_i + (1-\beta) \sum_{i=1}^{n} v_i \tilde{a}_i \quad (7)$$

where the $\tilde{a}_i$ are the argument variables represented in the form of FNs and $\beta \in [0, 1]$.

Note that sometimes, it is not clear how to reorder the arguments. Then, it is necessary to establish a criterion for comparing FNs. For simplicity, we recommend the following method. Select the FN with the highest value in its highest membership level, usually, when $\alpha = 1$. Note that if the membership level $\alpha = 1$ is an interval, then, we will calculate the average of the interval. If there is still a tie, then, we recommend the use of the average or a weighted average of the FN according to the interests of the decision maker.

Note that if the weighting vector of probabilities or WAs is not normalized, i.e., $P = \sum_{i=1}^{n} p_i \neq 1$, or $V = \sum_{i=1}^{n} v_i \neq 1$, then, the FPWA operator can be expressed as:

$$f(\tilde{a}_1, \ldots, \tilde{a}_n) = \frac{\beta}{P} \sum_{j=1}^{n} p_i \tilde{a}_i + \frac{(1-\beta)}{V} \sum_{i=1}^{n} v_i \tilde{a}_i \quad (8)$$

The FPWA is monotonic, commutative, bounded and idempotent. It is monotonic because if $\tilde{a}_i \geq u_i$, for all $\tilde{a}_i$, then, FPWA($\tilde{a}_1, \ldots, \tilde{a}_n$) ≥ FPWA($u_1, u_2 \ldots, u_n$). It is commutative because any permutation of the arguments has the same evaluation. That is, FPWA($\tilde{a}_1, \ldots, \tilde{a}_n$) = FPWA($u_1, u_2, \ldots, u_n$), where ($u_1, u_2, \ldots, u_n$) is any permutation of the arguments ($\tilde{a}_1, \ldots, \tilde{a}_n$). It is bounded because the FPWA aggregation is delimitated by the fuzzy minimum and the fuzzy maximum. That is, Min$\{\tilde{a}_i\}$ ≤ FPWA($\tilde{a}_1, \ldots, \tilde{a}_n$) ≤ Max$\{\tilde{a}_i\}$. It is idempotent because if $\tilde{a}_i = a$, for all $\tilde{a}_i$, then, FPWA($\tilde{a}_1, \ldots, \tilde{a}_n$) = $a$.

## 4 Families of FPWA Operators

Different families of FPWA operators are found by using a different manifestation in the weighting vectors of the probabilistic information and the weighted aggregation.

**Remark 1.** If $\beta = 0$, we get the fuzzy weighted average (FWA).

**Remark 2.** If $\beta = 1$, we get the fuzzy probabilistic approach.

**Remark 3.** If $p_i = 1/n$ and $v_i = 1/n$, for all $i$, then, we get the fuzzy average (FA). Note that the FA is also found if $\beta = 1$ and $p_i = 1/n$, for all $i$, and if $\beta = 0$ and $v_i = 1/n$, for all $i$.

**Remark 4.** If $v_i = 1/n$, for all $i$, then, we get the fuzzy probabilistic average.

**Remark 5.** If $p_i = 1/n$, for all $i$, then, we get the FWA with a probabilistic arithmetic mean.

**Theorem 1.** If the FNs are reduced to the usual exact numbers, then, the FPWA operator becomes the PWA operator [9].

**Proof.** Assume a TpFN = ($a_1$, $a_2$, $a_3$, $a_4$). If $a_1 = a_2 = a_3 = a_4$, then ($a_1$, $a_2$, $a_3$, $a_4$) = $a$, thus, we get the PWA operator.

**Remark 6.** In a similar way, we could develop the same proof for all the other types of FNs available in the literature [9].

**Theorem 2.** If the FNs are reduced to the interval numbers, then, the FPWA operator becomes the uncertain PWA (UPWA) operator [9].

**Proof.** Assume a TpFN = ($a_1$, $a_2$, $a_3$, $a_4$). If we only consider the points ($a_1$, $a_2$, $a_3$, $a_4$), then, the FN becomes an interval number (a quadruplet). Therefore, the FPWA operator becomes the UPWA operator.

**Remark 7.** In a similar way, we could develop the same proof for all the other types of FNs.

**Remark 8.** Note that similar analysis could be developed for considering situations when the FNs are representing linguistic variables, etc.

Table 1. Fuzzy payoff matrix.

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ |
|---|---|---|---|---|---|
| $A_1$ | (20,30,40) | (70,80,90) | (70,80,90) | (60,70,80) | (30,40,50) |
| $A_2$ | (50,60,70) | (60,70,80) | (40,50,60) | (60,70,80) | (50,60,70) |
| $A_3$ | (50,60,70) | (70,80,90) | (40,50,60) | (70,80,90) | (50,60,70) |
| $A_4$ | (70,80,90) | (60,70,80) | (70,80,90) | (10,20,30) | (60,70,80) |
| $A_5$ | (60,70,80) | (40,50,60) | (60,70,80) | (60,70,80) | (40,50,60) |

Table 3. Fuzzy aggregated results.

| | FA | FProb. | FWA | FPWA | PWA |
|---|---|---|---|---|---|
| $A_1$ | (50,60,70) | (49,59,69) | (53,63,73) | (51.4,61.4,71.4) | 61.4 |
| $A_2$ | (52,62,72) | (52,62,72) | (53,63,73) | (52.6,62.6,72.6) | 62.6 |
| $A_3$ | (56,66,76) | (56,66,76) | (58,68,78) | (57.2,67.2,77.2) | 67.2 |
| $A_4$ | (54,64,74) | (55,65,75) | (49,59,69) | (51.4,61.4,71.4) | 61.4 |
| $A_5$ | (52,62,72) | (54,64,74) | (54,64,74) | (54,64,74) | 64 |

## 5. Application in Decision Making

In the following, we present a numerical example of the new approach in a decision making problem about selection of monetary policies.

Note that similar problems could be developed in the selection of other policies such as fiscal and commercial policies. We analyze an economic problem about the monetary policy of a country. Assume the government of a country has to decide on the type of monetary policy to use the next year. They consider five alternatives:

- $A_1$ = Develop a strong expansive monetary policy.
- $A_2$ = Develop an expansive monetary policy.
- $A_3$ = Do not develop any change in the monetary policy.
- $A_4$ = Develop a contractive monetary policy.
- $A_5$ = Develop a strong contractive monetary policy.

In order to evaluate these policies, the government has brought together a group of experts. This group considers that the key factor is the economic situation of the world economy for the next period. They consider 5 possible states of nature that could happen in the future:

- $S_1$ = Very bad economic situation.
- $S_2$ = Bad economic situation.
- $S_3$ = Regular economic situation.
- $S_4$ = Good economic situation.
- $S_5$ = Very good economic situation.

The results of the available policies, depending on the state of nature $S_i$ and the alternative $A_k$ that the decision maker chooses, are shown in Table 1. Note that the results are FNs representing the benefits obtained by using each policy. The main advantage of using FNs with the FPWA operator is that we can represent the information in a more complete way because we can consider the minimum, the maximum and the most possible result.

In this problem, the experts assume the following probabilities representing the probability of occurrence of each state of nature: $P$ = (0.3, 0.2, 0.2, 0.2, 0.1). They assume that the WA, that represents the degree of importance of each state of nature, is: $V$ = (0.2, 0.2, 0.2, 0.3, 0.1). Note that the probabilistic information has an importance of 40% and the FWA an importance of 60%. For doing so, we will use Eq. (6) to calculate the FPWA aggregation. The results are shown in Table 2.

Table 2: FPWA weights

| | $\hat{v}_1$ | $\hat{v}_2$ | $\hat{v}_3$ | $\hat{v}_4$ | $\hat{v}_5$ |
|---|---|---|---|---|---|
| V* | 0.24 | 0.2 | 0.2 | 0.26 | 0.1 |

With this information, we can aggregate the expected results for each state of nature in order to make a decision. In Table 3, we present different results obtained by using different types of FPWA operators. Note that we also present the results obtained with the classical framework about using probabilities or WAs.

Note that we can also obtain these results by using Eq. (7). As we can see, in this example, the optimal choice is $A_3$.

## 6. Conclusions

We have presented a new approach that unifies the probability and the weighted average in the same formulation. Moreover, we have seen that this model is able to deal with uncertain information represented with FNs. We have called it the fuzzy probabilistic weighted averaging (FPWA) operator. We have also studied similar formulations although we have seen that the most complete one is the FPWA because it is able to unify both concepts considering the degree of importance that each concept has in the aggregation. We have also developed an application of the new approach in a decision making problem.

In future research, we expect to develop further extensions to this approach by using more general formulations and considering other characteristics in the problem.

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