Value function computation in fuzzy real options by differential evolution

Maria Letizia Guerra[1] Laerte Sorini[2] Luciano Stefanini[2]

1.Department of Mathematics for Economic and Social Sciences, University of Bologna, Italy
2. Department of Economics and Quantitative Methods, University of Urbino, Italy
Email: m.guerra@unibo.it, laerte.sorini@uniutb.it, luciano.stefanini@uniurb.it

Abstract

Real options are a typical framework in economics that involves uncertainty and can take advantage of a model of uncertainty that includes stochastic processes and fuzzy numbers; to perform the complete analysis with american type real options, we need to compute the fuzzy extension of the value function and this requires massive calculations.

A special version of the multiple population differential evolution algorithm is designed to compute the level-cuts of the fuzzy extension of the multidimensional real valued function of fuzzy numbers in the resulting optimization problems.

1 Introduction

Real options theory (ROT) is by now recognized as a most appropriate valuation technique for corporate investment decisions because of its distinctive ability to take into account management's flexibility to adapt ongoing projects in response to uncertain technological and market conditions. Dixit and Pindyck ([2]) develop a systematic treatment of ROT, providing the fundamentals of this method, using particularly dynamic programming and its connections with contingent claims analysis, and also emphasize the market implications of such valuation of investment decisions under uncertainty. Trigeorgis (see e.g. [7]) provides a taxonomy of real options that maps different categories of investments into the space of different types of financial options. We present a real options that will be evaluated within a fuzzy setting; more specifically, the present values of expected cash flows and expected costs are estimated by fuzzy numbers. To the best of our knowledge, such an approach has never been discussed in the literature, with the exception of Carlsson and Fuller [1], that interpret the possibility of making an investment decision in terms of a European option, while we use an American option.

A special version of the multiple population differential evolution algorithm is designed to compute the level-cuts of the fuzzy extension of the multidimensional real valued function of fuzzy numbers in the resulting optimization problems. We perform some computational experiments connected with the option to defer investment, that is an American call option on the present value of the completed expected cash flows with the exercise price equal to the required outlay.

2 Basic fuzzy numbers and fuzzy arithmetic

Fuzzy numbers are a very powerful and flexible way to describe uncertainty or possibilistic values for given variables for which a precise quantification is not possible or one is interested in evaluating the effects of variations around a specified value (see [3]). A wide class of fuzzy numbers with the core at $a \in \mathbb{R}$ is obtained by considering its membership function $\mu : \mathbb{R} \longrightarrow [0,1]$ such that, denoting $[a^-, a^+]$ the interval representing the support (corresponding to the membership level $\alpha = 0$),

$$\mu(x) = \begin{cases} L(x) & if & a^- \le x \le a \\ R(x) & if & a \le x \le a^+ \\ 0 & otherwise & \end{cases} \quad \text{for } x \in \mathbb{R} \tag{1}$$

where $L(x)$ is an increasing function with $L(a^-) = 0$, $L(a) = 1$ and $R(x)$ is a decreasing function with $R(a) = 1$, $R(a^+) = 0$.

For values of $\alpha \in]0,1]$, the $\alpha - cut$ is defined to be the compact interval $[u]_\alpha = \{x|\mu(x) \ge \alpha\}$ and the support is $[u]_0 = cl\{x|\mu(x) > 0\}$ ($cl(A)$ is the closure of set A); denote $[u]_\alpha = [u^-_\alpha, u^+_\alpha]$ for $\alpha \in [0,1]$. The support of u is the interval $[u^-_0, u^+_0]$ and the core is $[u^-_1, u^+_1]$. If $u^-_1 < u^+_1$ we have a fuzzy interval and if $u^-_1 = u^+_1$ we have a fuzzy number. We refer to functions $u^-_{(.)}$ and $u^+_{(.)}$ as the lower and upper branches on u, respectively.

To model the monotonic branches u^-_α and u^+_α we start with two increasing shape functions p^-, p^+ such that $p(0) = 0$ and $p(1) = 1$ with the four numbers $u^-_0 \le$

978-0-7695-3872-3/09 $26.00 © 2009 IEEE
DOI 10.1109/ISDA.2009.232

$u_1^- \leq u_1^+ \leq u_0^+$ defining the support $\left[u_0^-, u_0^+\right]$ and the core $\left[u_1^-, u_1^+\right]$ and we define

$$\begin{aligned} u_\alpha^- &= u_0^- + (u_1^- - u_0^-)p^-(\alpha) \text{ and} \\ u_\alpha^+ &= u_0^+ + (u_1^+ - u_0^+)p^+(\alpha) \text{ for all } \alpha \in [0,1]. \end{aligned} \tag{2}$$

The two shape functions p^- and p^+ are selected in a family of parametrized monotonic functions.

The simplest representation is obtained on the trivial decomposition of the interval $[0,1]$, with $\alpha_0 = 0, \alpha_1 = 1$. In this simple case, u can be represented by a vector of 8 components

$$u = (u_0^-, \delta u_0^-, u_0^+, \delta u_0^+; u_1^-, \delta u_1^-, u_1^+, \delta u_1^+) \tag{3}$$

where $u_0^-, \delta u_0^-, u_1^-, \delta u_1^-$ are used for the lower branch u_α^-, and $u_0^+, \delta u_0^+, u_1^+, \delta u_1^+$ for the upper branch u_α^+. The parameters $\delta u_0^-, \delta u_1^- \geq 0$ are the first derivatives (slopes) of u_α^- with respect to α at $\alpha = 0$ and $\alpha = 1$; the parameters $\delta u_0^+, \delta u_1^+ \leq 0$ are the first derivatives of u_α^+ at $\alpha = 0$ and $\alpha = 1$, respectively. More generally, if the functions u_α^- and u_α^+ (we suppose for simplicity that they are differentiable) and the corresponding slopes are known on a decomposition $0 = \alpha_0 < \alpha_1 < ... < \alpha_N = 1$ of the interval $[0,1]$, i.e. if we know the values $u_{\alpha_i}^- = u_i^-$, $u_{\alpha_i}^+ = u_i^+$ and the first derivatives $\delta u_i^- \geq 0$, $\delta u_i^+ \leq 0$, then we use equations (2) piecewise on each subinterval $[\alpha_{i-1}, \alpha_i]$ for $i = 1, 2, ..., N$ and obtain the general parametrization

$$u = (\alpha_i; u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,1,...,N}. \tag{4}$$

We call (4) the LU parametrization (or LU-fuzzy representation) of u. Details can be found in [5] and [6].

3 Differential Evolution algorithms for fuzzy arithmetic

We adopt an algorithmic approach to describe the application of differential evolution methods to calculate the fuzzy extension of multivariable function, associated to the LU parametrization.

Let $v = f(u_1, u_2, ..., u_n)$ denote the fuzzy extension of a continuous function f in n variables; it is well known that the fuzzy extension of f to normal upper semicontinuous fuzzy intervals (with compact support) has the level-cutting commutative property (see [3]), i.e. the $\alpha - cuts$ $v_\alpha = [v_\alpha^-, v_\alpha^+]$ of v are the images of the $\alpha - cuts$ of $(u_1, u_2, ..., u_n)$ and are obtained by solving the box-constrained optimization problems

$$\left\{ \begin{array}{l} v_\alpha^- = \min \left\{ \begin{array}{c} f(x_1, ..., x_n)| \\ x_k \in [u_{k,\alpha}^-, u_{k,\alpha}^+], \end{array} \right\} \\ v_\alpha^+ = \max \left\{ \begin{array}{c} f(x_1, ..., x_n)| \\ x_k \in [u_{k,\alpha}^-, u_{k,\alpha}^+], \end{array} \right\}. \end{array} \right. \tag{5}$$

We will consider differentiable functions f. If $u_k = (u_{k,i}^-, \delta u_{k,i}^-, u_{k,i}^+, \delta u_{k,i}^+)_{i=0,1,...,N}$ are the LU-fuzzy representations of the n input quantities and

$$v = (v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,1,...,N}, \tag{6}$$

then the $\alpha - cuts$ of v are obtained by solving (5).

For each $\alpha = \alpha_i$, $i = 0, 1, ..., N$ the $\min\{\}$ and the $\max\{\}$ can occur either at a point whose components $x_{k,i}$ are internal to the corresponding intervals $[u_{k,i}^-, u_{k,i}^+]$ or are coincident with one of the extremal values; denote by $\widehat{x}_i^- = (\widehat{x}_{1,i}^-, ..., \widehat{x}_{n,i}^-)$ and $\widehat{x}_i^+ = (\widehat{x}_{1,i}^+, ..., \widehat{x}_{n,i}^+)$ the points where the min and the max take place; then $v_i^- = f(\widehat{x}_{1,i}^-, ..., \widehat{x}_{n,i}^-)$ and $v_i^+ = f(\widehat{x}_{1,i}^+, ..., \widehat{x}_{n,i}^+)$ and the slopes δv_i^-, δv_i^+ are computed (as f is differentiable) by

$$\begin{aligned} \delta v_i^- &= \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^- = u_{k,i}^-}}^{n} \frac{\partial f(\widehat{x}_{1,i}^-, ..., \widehat{x}_{n,i}^-)}{\partial x_k} \delta u_{k,i}^- \\ &+ \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^- = u_{k,i}^+}}^{n} \frac{\partial f(\widehat{x}_{1,i}^-, ..., \widehat{x}_{n,i}^-)}{\partial x_k} \delta u_{k,i}^+ \end{aligned} \tag{7}$$

$$\begin{aligned} \delta v_i^+ &= \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^+ = u_{k,i}^-}}^{n} \frac{\partial f(\widehat{x}_{1,i}^+, ..., \widehat{x}_{n,i}^+)}{\partial x_k} \delta u_{k,i}^- \\ &+ \sum_{\substack{k=1 \\ \widehat{x}_{k,i}^+ = u_{k,i}^+}}^{n} \frac{\partial f(\widehat{x}_{1,i}^+, ..., \widehat{x}_{n,i}^+)}{\partial x_k} \delta u_{k,i}^+. \end{aligned} \tag{8}$$

The idea of DE to find *min* or *max* of $\{f(x_1, ..., x_n)|(x_1, ..., x_n) \in \mathbb{A} \subset \mathbb{R}^n\}$ is simple: start with an initial "population" $x^{(1)} = (x_1, ..., x_n)^{(1)}, ...,$ $x^{(p)} = (x_1, ..., x_n)^{(p)} \in \mathbb{A}$ of p feasible points for each generation (i.e. for each iteration) to obtain a new set of points by recombining randomly the individuals of the current population and by selecting the best generated elements to continue in the next generation. The initial population is chosen randomly and should try to cover uniformly the entire parameter space.

Denote by $x^{(k,g)}$ the $k-$th vector of the population at iteration (generation) g and by $x_j^{(k,g)}$ its $j-$th component $(j = 1, ..., n)$.

At each iteration, the method generates a set of candidate points $y^{(k,g)}$ to substitute the elements $x^{(k,g)}$ of the current population, if $y^{(k,g)}$ is better.

To generate $y^{(k,g)}$ two operations are applied: recombination and crossover.

A typical recombination operates on a single component $j \in \{1, ..., n\}$ and generates a new perturbed vector of the form $v_j^{(k,g)} = x_j^{(r,g)} + \gamma[x_j^{(s,g)} - x_j^{(t,g)}]$, where $r, s, t \in \{1, 2, ..., p\}$ are chosen randomly and $\gamma \in]0, 2]$ is a constant (eventually chosen randomly for the current iteration) that controls the amplification of the variation.

The potential diversity of the population is controlled by a crossover operator, that construct the candidate $y^{(k,g)}$ by crossing randomly the components of the perturbed vector $v_j^{(k,g)}$ and the old vector $x_j^{(k,g)}$:

$$y_j^{(k,g)} = \begin{cases} v_j^{(k,g)} & \text{if } j \in \{j_1, j_2, ..., j_h\} \\ x_j^{(k,g)} & \text{if } j \notin \{j_1, j_2, ..., j_h\} \end{cases}$$

where h is a random integer between 0 and n (it is 0 with probability q) and $j_1, j_2, ..., j_h$ are random components if h is not 0; so, the components of each individual of the current population are modified to $y_j^{(k,g)}$ by a given probability q. Typical values are $\gamma \in [0.2, 0.95]$, $q \in [0.7, 1.0]$ and $p \geq 5n$ (the higher p, the lower γ).

The candidate $y^{(k,g)}$ is then compared to the existing $x^{(k,g)}$ by evaluating the objective function at $y^{(k,g)}$: if $f(y^{(k,g)})$ is better than $f(x^{(k,g)})$ then $y^{(k,g)}$ substitutes $x^{(k,g)}$ in the new generation $g+1$, otherwise $x^{(k,g)}$ is retained.

To take into account the particular nature of our problem, we modify the basic procedure and examine two different strategies.

Let $[u_{k,i}^-, u_{k,i}^+]$, $k = 1, 2, ..., n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be given; we have to find v_i^- and v_i^+ according to (5) for $i = 0, 1, ..., N$. The slope parameters δv_i^-, δv_i^+ are computed by (7)and (8).

The first strategy is implemented in algorithm 1. Function $ran(0,1)$ generates a random uniform number in [0,1].

SPDE (Single Population DE procedure): start with the $(\alpha = 1) - cut$ back to the $(\alpha = 0) - cut$ so that the optimal solutions at a given level can be inserted into the "starting" populations of lower levels; use two distinct populations and perform the recombinations such that, during generations, one of the populations specializes to find the minimum and the other to find the maximum.

MPDE (Multi Populations DE procedure): use $2(N+1)$ populations to solve simultaneously all the box-constrained problems (5); $N+1$ populations specialize for the min and the others for the max and the current best solution for level α_i is valid also for levels $\alpha_0, ..., \alpha_{i-1}$. The details of MPDE are descibed in the following algorithm.

Algorithm MPDE: (Frame of MPDE).
Choose $p \approx 5n$, $g_{\max} \approx 500$, q and γ.
Select $(x_1^{(l,i)}, ..., x_n^{(l,i)})$, $x_k^{(l,i)} \in [u_{k,i}^-, u_{k,i}^+]$
$\forall k$, $l = 1, ..., 2p$, $i = 0, 1, ..., N$
let $y^{(l,i)} = f(x_1^{(l,i)}, ..., x_n^{(l,i)})$
let $v_i^- = \min \{y^{(l,j)} | j = 0, ..., i, \forall l\}$
let $v_i^+ = \max \{y^{(l,j)} | j = 0, ..., i, \forall l\}$
let $\widehat{x}_i^-, \widehat{x}_i^+ \in R^n$ the points where v_i^-, v_i^+ are taken
for $g = 1, 2, ..., g_{\max}$
  (up to $g_{\max}$ generations or other stopping rule)
  for $i = N, N-1, ..., 0$
    for $l = 1, 2, ..., p$
      select (randomly) $r, s, t \in \{1, 2, ..., p\}$
      and $k^* \in \{1, 2, ..., n\}$
      for $k = 1, 2, ..., n$
        if ($k = k^*$ or $ran(0,1) < q$) then
          $x'_k = x_k^{(r,i)} + \gamma[x_k^{(s,i)} - x_k^{(t,i)}]$
          $x''_k = x_k^{(p+r,i)} + \gamma[x_k^{(p+s,i)} - x_k^{(p+t,i)}]$
          ensure $u_{k,i}^- \leq x'_k, x''_k \leq u_{k,i}^+$
        else
          $x'_k = x_k^{(l,i)}$, $x''_k = x_k^{(p+l,i)}$
        endif
      end
      let $y' = f(x'_1, ..., x'_n)$ and $y'' = f(x''_1, ..., x''_n)$;
      if $y' < y^{(l,i)}$ (population for min)
        substitute $(x_1, ..., x_n)^{(l,i)}$ with $(x'_1, ..., x'_n)$
      if $y'' > y^{(p+l,i)}$ (population for max)
        substitute $(x_1, ..., x_n)^{(p+l,i)}$ with $(x''_1, ..., x''_n)$
      if y' or y'' are better
        update values $\{v_j^-, v_j^+, \widehat{x}_j^-, \widehat{x}_j^+ | j = 0, ..., i\}$
      endif
    end
  end
end

In our case, as we have simple box-constraints, it is easy to produce feasible starting populations, as we have to generate random numbers $x_j^{(k,0)}$ between the lower $u_{j,i}^-$ and the upper $u_{j,i}^+$ values.

During the iterations, we use a variant of the method above, where the $y^{(k,g)}$ are progressively forced to be feasible or with small infeasibilities and a penalty is assigned to infeasible values:

(i) modify $y_j^{(k,g)}$ to fit $[u_{j,i}^- - \frac{\varepsilon}{g^2}, u_{j,i}^+ + \frac{\varepsilon}{g^2}]$, $j = 1, 2, ..., n$ with small $\varepsilon \sim 10^{-2}(u_{j,i}^+ - u_{j,i}^-)$, so that the eventual infeasibilities decrease rapidly during the generation process;

(ii) if the candidate point $y^{(k,g)}$ is infeasible and has a value $f(y^{(k,g)})$ better than the current best feasible value $f(x^{(best,g)})$ then a penalty is added and the value of $y^{(k,g)}$ is elevated to $f(x^{(best,g)}) + \varepsilon'$ (for the min problems) or reduced to $f(x^{(best,g)}) - \varepsilon'$ (for the max problem), being ε' $\sim 10^{-3}$ a small positive number.

To decide that a solution is found, we use the following simple rule: choose a fixed tolerance $tol \sim 10^{-3}, 10^{-4}$ and a number $\widehat{g} \sim 20, 30$ of generations; if for $\widehat{g}$ subsequent iterations all the values v_i^- and v_i^+ are changed less than tol, then the procedure stops and the found solution is assumed to be optimal. In any case, no more than 500 iterations are performed (but this limit was never reached during the computations).

4 Fuzziness in option to defer investment

The *option to defer investment* is an American call option on the present value of the completed expected cash flows with the exercise price being equal to the required outlay. A project that can be postponed allows learning more about potential project outcomes before making a commitment (see the seminal contribution by McDonald and Siegel [4]). A firm is supposed to consider the following investment opportunity: at any time t the firm can pay some estimated cost K to install an investment project whose expected future net cash flows conditional on undertaking the project have an estimated present value Π. If $V = V(\Pi)$ is the option value then the following second order ordinary differential equation holds:

$$\frac{1}{2}\sigma^2\Pi^2 V''(\Pi) + \mu\Pi V'(\Pi) - rV = 0$$

for $\Pi < \Pi^*$ with the initial condition $V(0) = 0$ and smooth-pasting $V(\Pi^*) = \Pi^* - K, V'(\Pi^*) = 1$. The solution is

$$\begin{cases} \Pi^* = K\frac{\phi}{\phi-1} \\ V(\Pi) = (\Pi^* - K)\left(\frac{\Pi}{\Pi^*}\right)^{\phi} \end{cases} \tag{9}$$

with $\phi = \frac{1}{2} - \frac{\mu}{\sigma^2} + \left(\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}\right)^{\frac{1}{2}} > 1$. As soon as Π reaches the threshold value Π^*, the firm finds it optimal to invest (case of the option to defer investment) or disinvest and liquidate (case of the option to abandon). Thus, the decision is based on the threshold value, which depends on all the parameters of the model.

Our formalization of the valuation of real options schedules the presence of fuzziness in three fundamental steps: (a) in the stochastic differential equation driving the dynamics of Π, we assume μ, σ and the initial value of Π to be fuzzy; (b) the valuation function of the option depends not only on Π, σ and μ but also on r and K, which we assume to be fuzzy too; (c) as fuzziness affects the crucial threshold value Π^*, the process $\{\Pi_t, t \geq 0\}$ is assumed to be a fuzzy stochastic process and Π^* is itself a fuzzy quantity; correspondingly, $V(\Pi^*)$ is fuzzy and its membership is to be computed.

The extension principle is then applied to obtain the fuzzy Π^* and $V(\Pi^*)$ from the exact solutions given in equation (9). In the formulae (6) and (7)-(8) the vector $\widehat{x}_i$ is equal to $(\widehat{\mu}_i, \widehat{\sigma}_i, \widehat{r}_i, \widehat{K}_i)$ and some of the partial derivatives that define the slopes of the representation are nothing else than the first order Greeks, in particular, $\frac{\partial f(\widehat{\mu}_i,\widehat{\sigma}_i,\widehat{r}_i,\widehat{K}_i)}{\partial\sigma}$ is the Vega and $\frac{\partial f(\widehat{\mu}_i,\widehat{\sigma}_i,\widehat{r}_i,\widehat{K}_i)}{\partial r}$ is the Rho.

The degree of the uncertainty and the way in which it is spread from the model, play a central role in the analysis of the real option. The nonlinearities entering in the definition of $V(\Pi)$ in (9) are the main cause of such effects and they can propagate or contract uncertainty. It is very important to perceive the magnitude and the type of these effects. In particular we are interested in the analysis of how the various kinds of uncertainties inserted into the parameters will produce the corresponding uncertainties in Π^*, $V^* = V(\Pi^*)$.

As soon as information (on μ, σ, r, K) is modelled by fuzzy numbers, Π^* and V^* also become fuzzy and are represented by $\alpha - cuts$ $\left[\Pi_\alpha^{*-}, \Pi_\alpha^{*+}\right]$ and $\left[V_\alpha^{*-}, V_\alpha^{*+}\right]$ for all degrees of possibility α. The maximal uncertainty corresponds to the supports at $\alpha = 0$, given by the intervals $\left[\Pi_0^{*-}, \Pi_0^{*+}\right]$ and $\left[V_0^{*-}, V_0^{*+}\right]$ for Π^* and V^* respectively.

Due to the nonlinearity of Π^* and V^*, the $\alpha - cuts$ are not necessarily symmetric and, for a given uncertainty on the input values μ, σ, r and K, they have different left and right variations. Let $\widehat{\Pi}^*$ and $\widehat{V}^*$ denote the values of Π_α^* and V_α^* corresponding to $\alpha = 1$. It is immediate to argue that V^* is symmetric if and only if $\Delta V_\alpha^{*+} = \Delta V_\alpha^{*-}, \forall\alpha \in [0, 1[$ where

$$\Delta V_\alpha^{*+} = V_\alpha^{*+} - \widehat{V}^*, \quad \Delta V_\alpha^{*-} = \widehat{V}^* - V_\alpha^{*-}.$$

The quantity ΔV_α^{*+} represents the possible increase in $\widehat{V}^*$ due to uncertainty and analogously, ΔV_α^{*-} measures the possible decrease. The same argument can be applied to Π_α^* and $\widehat{\Pi}^*$, defining the quantities $\Delta\Pi_\alpha^{*+} = \Pi_\alpha^{*+} - \widehat{\Pi}^*$ and $\Delta\Pi_\alpha^{*-} = \widehat{\Pi}^* - \Pi_\alpha^{*-}$.

An index that measures the propagation of uncertainty on the right and left sides is the following asymmetry ratio S (such that $S \geq 0$); for a given value of α we can compute:

$$S_\alpha = \frac{\Delta\Pi_\alpha^{*+}}{\Delta\Pi_\alpha^{*-}}. \tag{10}$$

If $\alpha = 1$ we set $S_1 = 1$; if α decreases to zero, both numerator and denominator will increase with different magnitudes reported by their ratio: when $S_\alpha > 1$, it means that, for the given level α of uncertainty, the right semi-interval is larger than the left one, in other words it is more possible to obtain bigger values than the crisp one instead of smaller; when $S_\alpha < 1$, the reverse holds.

5 Computational experiments

We test the fuzziness effect in the option to defer investment by running several computational experiments; the fuzzy version of the indicated parameters (say θ) is assumed as triangular (linear shaped) symmetric fuzzy numbers, centered at the crisp values and with the support being the interval $[\theta - 0.1\theta, \theta + 0.1\theta]$, corresponding to a symmetric uncertainty of 10%. To analyze the effect of the uncertainty on the output variables, we compute their membership functions for $\alpha - levels$ with $\alpha = 1$ (the crisp level), $\alpha = 0.75$

(corresponding to the uncertainty of 2.5% in the parameters), $\alpha = 0.5$ (corresponding to the uncertainty of 5% in the parameters), $\alpha = 0.25$ (corresponding to the uncertainty of 7.5% in the parameters), $\alpha = 0$ (corresponding to the uncertainty of 10% in the parameters).

The robustness of the fuzzy model for the option to defer investment is presented with two sets of real data that we call, for short, Test1 and Test2, referring to two different industrial sectors. *Test1* refers to an investment decision in the human genome sciences project (HGSI) whose data are taken from the Human Genome project database. *Test2* refers to an investment decision in a big infrastructure, that is the Eurotunnel project. The values of the parameters μ, σ, r and K are the following:

	Test1	Test2
μ	0.01	0.025
σ	0.048	0.183
r	0.044	0.06
K	704.9	8865

We show the shape of V^* in the two cases of real data and the preliminary consideration attains the fact that it exists a uniformity in the results about the ROT behavior even if the cases under consideration belong to deeply different industrial areas. In tables concerning the behavior of Π^* we report three different cases that we will denote as: *Allfuzzy*, when the parameters μ, σ, r and K are fuzzy; *Kcrisp*, when μ, σ, r are fuzzy and K is crisp and finally *Kfuzzy*, when μ, σ, r are crisp and K is the unique source of uncertainty.

5.0.1 Results for Test1

As expected, the greatest uncertainty in Π^* occurs in the Allfuzzy case, when all the fuzzy quantities are considered to be fuzzy; but it is interesting to observe that in the Kcrisp case the generated uncertainty is less then in the Kfuzzy case, i.e. the uncertainty in the values of only K produces more uncertainty on Π^* then the uncertainty in the values of μ, σ and r.

Table 1, Table 2 and Table 3 report values of the $\alpha - cut$ for Test1 in the Allfuzzy, Kcrisp and Kfuzzy case respectively.

Table 1

α	Π^-	Π^+	S
1.0	994.28	994.28	1
0.75	953.16	1037.56	1.05
0.5	913.97	1083.26	1.11
0.25	876.50	1131.69	1.17
0	840.58	1183.25	1.23

Results for Test1, Allfuzzy case

Table 2

α	Π^-	Π^+	S
1.0	994.28	994.28	1
0.75	977.60	1012.25	1.08
0.5	962.07	1031.67	1.16
0.25	947.56	1052.74	1.25
0.0	933.98	1075.68	1.35

Test1, Kcrisp case

Table 3

α	Π^-	Π^+
1.0	994.28	994.28
0.75	969.43	1019.14
0.5	944.57	1043.99
0.25	919.71	1068.86
0	894.86	1093.71

Results for Test1, Kfuzzy case.

Observe that in the Kfuzzy case (Table 3) the threshold value Π^* displays a symmetric shape in all analyzed projects because Π^* depends linearly on K ($S = 1$). In the Allfuzzy and Kcrisp cases, instead, we can observe an asymmetric pattern, due to the nonlinear dependence of Π^* with respect to the other variables.

At level 0.5 the average values are 998.615 in Allfuzzy and 996.87 in Kcrisp, which are larger than the crisp value 994.28. Since on average the fuzzy threshold value is larger then without fuzziness, just considering the crisp value the decision to invest would be too early. Figure 4 shows the graphical behavior of the fuzzy function $V(\Pi)$ in the Allfuzzy case; the little crosses point the optimal values of Π corresponding to the levels of Π^* for $\alpha = 0, 0.25, 0.5, 0.75, 1$.

It is evident that fuzziness implies a certain degree of freedom in the choice of Π^*. Figures 1 and 2 illustrate $V(\Pi)$ as a fuzzy function (a "sequence" of fuzzy numbers).

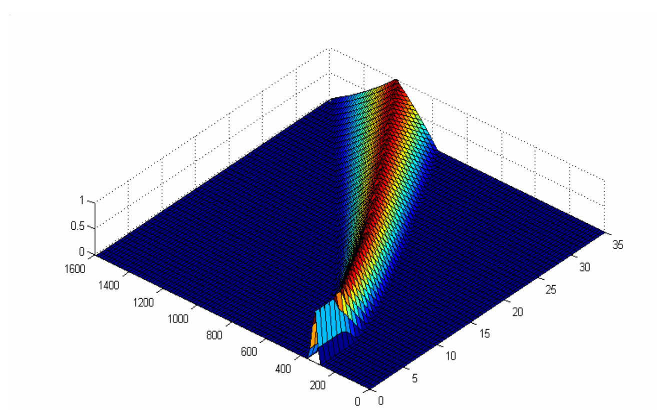

Figure 1. A case of fuzzy function $V(\Pi)$

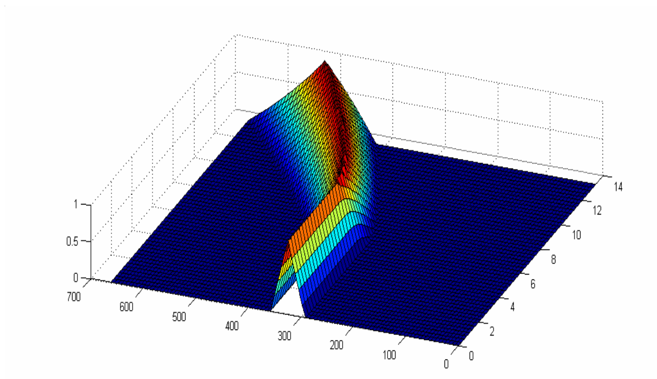

Figure 2. A second case of fuzzy function $V(\Pi)$

Obtaining $V(\Pi)$ is the part where massive computation (i.e. repeated application of the DE algorithm) is required.

5.0.2 Results for Test2

Table 4 and Table 5 report the values of the $\alpha - cut$ of Π^* in the Allfuzzy and Kcrisp case for Test2. Again the biggest uncertainty occurs in the Allfuzzy case, when all the quantities are fuzzy; but we observe that in the Kcrisp case (dotted line), when μ, σ, r are the sources of uncertainty and K is the unique crisp value, the uncertainty in Π^* is bigger then in the Kfuzzy case, i.e. the same level of uncertainty in K produses less uncertainty on Π^* then the uncertainty in the other parameters. With respect to Test1, there is here un inversion.

Table 4

α	Π^-	Π^+	S
1.0	22249.62	22249.62	1
0.75	20686.18	23997.55	1.12
0.5	19277.60	25967.74	1.25
0.25	18000.33	28209.16	1.40
0	16835.43	30786.48	1.58

Result for Test2, Allfuzzy case.

Table 5

α	Π^-	Π^+	S
1.0	22249.62	22249.62	1
0.75	21216.59	23412.24	1.12
0.5	20292.21	24731.18	1.27
0.25	19459.82	26241.08	1.43
0	18706.04	27987.71	1.62

Results for Test2, Kcrisp case

If we compute again the average values at level 0.5, they are 22622.67 in Allfuzzy and 22511.695 in Kcrisp, which are larger than the crisp value 22249.62. It follows that in the Test2 project it is confirmed the suggestion to wait for the decision to invest.

The parameter S is again always bigger than 1, indicating that bigger values are more possible than smaller values. This aspect is recurring in all simulations and it probably derives from the shape of the function V that assumes bigger values always on the right part of its graph.

6 Concluding remarks

Some further considerations concerning the $\alpha - cut$ values in all the data set enable us to state that our model allows us to describe how the investment decision is actually affected by a perceived increase in "fuzziness". For a pessimistic (optimistic) firm an increase in fuzziness decreases (increases) the perceived value of the project in comparison with the crisp value. On average - for most decision makers- an increase in fuzziness has a positive impact on the investment opportunity, i.e. it increases the perceived value of the project. As a consequence, the decision to invest is delayed in comparison with the absence of fuzziness. However, for pessimistic decision-makers imprecise information about the project value becomes available over time, which makes waiting with investment less valuable. Thus, for pessimistic firms higher fuzziness erodes the subjective value of the investment opportunity. Notice that this result is in keeping with the literature on real options and ambiguity aversion. It contrasts with the impact of volatility in the standard real option theory.

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