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Bi-symmetrically Weighted Trapezoidal Approximations of Fuzzy Numbers

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***Abstract*—Trapezoidal approximation of fuzzy numbers preserving the expected interval is considered. A general problem of the trapezoidal approximation of fuzzy numbers with respect to the distance based on bi-symmetrical weighted functions is solved. A practical algorithm for constructing approximation operator is given.**

***Keywords*-expected interval; fuzzy number; trapezoidal approximation; weighted distance**

I. Introduction

Operations on fuzzy numbers depend strongly on the shape of the membership functions under study. Complicated membership functions may cause difficulties in management and processing imprecise data modelled by fuzzy objects. This is the reason that approximation methods for simplifying original fuzzy numbers are of interest.

Interval approximations of fuzzy numbers were considered by [4], [9], [10]. Recently many researchers are interested in trapezoidal approximations (see, e.g. [1], [2], [3], [5], [13], [14], [11], [12], [18]). The main reason for such approximation is that a trapezoidal fuzzy number is completely represented by four real numbers only and hence it is easier to process and manage. Since even trapezoidal approximation might be performed in many ways, a list of criteria which a desired approximation operator should possess was formulated in [13].

The trapezoidal approximation under weighted distance was considered in [21] (some gaps in that paper were removed in [19] and [20] where so-called extended trapezoidal fuzzy numbers were introduced and applied). The most common weighing applied to fuzzy numbers is a linearly increasing one (see, e.g. [5], [21]). However, it seems that a slightly modified weighing might be more natural and more convenient in many situations. Two examples of so-called bi-symmetrical weighted functions were introduced and examined in [15]. In the present paper we generalize these results and show the nearest trapezoidal approximation operator preserving the expected interval, with respect to the distance based on any regular bi-symmetrical weighted function.

The paper is organized as follows. In Sec. 2 we recall some basic notions related to fuzzy numbers. In Sec. 3 we present a general idea of the trapezoidal approximation and related problems. Finally, in Sec 4, we show the main result, i.e. a theorem and a practical algorithm for constructing trapezoidal approximation operators based on the weighted distance utilizing regular bi-symmetrical weighted functions.

II. Basic concepts and notation

Let A denote a fuzzy number, i.e. such fuzzy subset A of the real line $\mathbb{R}$ with membership function $\mu_A : \mathbb{R} \to [0,1]$ which is (see [6]): normal (i.e. there exist an element x_0 such that $\mu_A(x_0) = 1$), fuzzy convex (i.e. $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \mu_A(x_1) \wedge \mu_A(x_2)$, $\forall x_1, x_2 \in \mathbb{R}$, $\forall \lambda \in [0,1]$), μ_A is upper semicontinuous, supp$A$ is bounded, where supp$A = cl(\{x \in \mathbb{R} : \mu_A(x) > 0\})$, and cl is the closure operator. A space of all fuzzy numbers will be denoted by $\mathbb{F}(\mathbb{R})$.

Moreover, let $A_\alpha = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$, $\alpha \in (0,1]$, denote an α-cut of a fuzzy number A. As it is known, every α-cut of a fuzzy number is a closed interval, i.e. $A_\alpha = [A_L(\alpha), A_U(\alpha)]$, where $A_L(\alpha) = \inf\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$ and $A_U(\alpha) = \sup\{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$.

The expected interval $EI(A)$ of a fuzzy number A is given by (see [7], [16])

$$\begin{aligned} EI(A) &= [EI_L(A), EI_U(A)] \\ &= \left[\int_0^1 A_L(\alpha)d\alpha, \int_0^1 A_U(\alpha)d\alpha\right]. \end{aligned} \tag{1}$$

The middle point of the expected interval

$$EV(A) = \frac{1}{2}\left(\int_0^1 A_L(\alpha)d\alpha + \int_0^1 A_U(\alpha)d\alpha\right) \tag{2}$$

is called the *expected value* of a fuzzy number and represents the typical value of the fuzzy number A (see [7], [16]).

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Another useful parameter characterizing a fuzzy number is called the *width* (see [4]) and is defined by

$$w(A) = \int_{-\infty}^{\infty} \mu_A(x)dx = \int_0^1 (A_U(\alpha) - A_L(\alpha))d\alpha. \quad (3)$$

For two arbitrary fuzzy numbers A and B with α-cuts $[A_L(\alpha), A_U(\alpha)]$ and $[B_L(\alpha), B_U(\alpha)]$, respectively, the quantity

$$d(A,B) = \left(\int_0^1 [A_L(\alpha) - B_L(\alpha)]^2 d\alpha + \int_0^1 [A_U(\alpha) - B_U(\alpha)]^2 d\alpha \right)^{1/2} \quad (4)$$

is the distance between A and B (for more details we refer the reader to [8]).

III. Trapezoidal approximations of fuzzy numbers

Suppose that for some reasons or just for simplicity we need a suitable approximation of a fuzzy number under study. A sufficiently effective simplification of the membership function characterizing a fuzzy number can be reached by the piecewise linear curves leading to triangle, trapezoidal or orthogonal membership curves. These three mentioned shapes are particular cases of the trapezoidal membership function. A family of all trapezoidal fuzzy numbers will be denoted by $\mathbb{F}^T(\mathbb{R})$. Thus, further on we will consider just the trapezoidal approximation of fuzzy numbers. It means that we will substitute given fuzzy number A by the trapezoidal fuzzy number $T(A)$, i.e. by a fuzzy number with a following membership function

$$\mu_{T(A)}(x) = \begin{cases} 0 & \text{if} \quad x < t_1, \\ \frac{x-t_1}{t_2-t_1} & \text{if} \quad t_1 \le x < t_2, \\ 1 & \text{if} \quad t_2 \le x \le t_3, \\ \frac{t_4-x}{t_4-t_3} & \text{if} \quad t_3 < x \le t_4, \\ 0 & \text{if} \quad t_4 < x, \end{cases} \quad (5)$$

where $t_1, t_2, t_3, t_4 \in \mathbb{R}$ and $t_1 \le t_2 \le t_3 \le t_4$.

One can do this in many ways so we need some additional constraints which guarantee that our approximation would be reasonable. It seems that the most natural idea is to construct $T(A)$ to be the closest to the original fuzzy number A with respect to given distance d. We may also consider other requirements which warrant that our approximation would possess some desired properties, like preservation of some fixed parameters or relations, continuity, etc. This problem was considered by many authors (see, e.g. [1], [2], [3], [5], [13], [14], [11], [12], [18], [19]). For example, it was suggested in [13] to consider *the nearest trapezoidal approximation operator preserving the expected interval*, i.e. the approximation operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ which produces a trapezoidal fuzzy number $T(A)$ that is the closest with respect to distance (4), i.e.

$$d(A,T(A)) = \left(\int_0^1 [A_L(\alpha) - T(A)_L(\alpha)]^2 d\alpha + \int_0^1 [A_U(\alpha) - T(A)_U(\alpha)]^2 d\alpha \right)^{1/2} \quad (6)$$

to given original fuzzy number A among all trapezoidal fuzzy numbers having identical expected interval as the original one, i.e. satisfying a following condition

$$EI(T(A)) = EI(A). \quad (7)$$

It is worth noting the invariance of the expected interval assures many other useful properties, like preservation of the expected value and of the width, order invariance with respect to some preference fuzzy relations and correlation invariance. For more details we refer the reader to [13] where the broad list of desired requirements that the approximation operator should possess is also given (the research on this operator was continued in [3], [14], [11], [12]).

By (5) any trapezoidal fuzzy number is completely described by four real numbers that are borders of its support and core. So in trapezoidal approximation the goal reduces to finding such real numbers $t_1 \le t_2 \le t_3 \le t_4$ that characterize $T(A) = T(t_1, t_2, t_3, t_4)$. Let us mention that mathematical formulae for these points corresponding to the nearest trapezoidal approximation operator preserving the expected interval are given in [11].

In some situations other distances than (6) might be more suitable. Using (6) all α-cuts are treated evenly. This feature is sometimes criticized by authors who claim that elements belonging to α_1-cut should be treated with the higher attention that those from α_2-cut if $\alpha_1 > \alpha_2$ because the membership degree for the first group is higher and so they are less uncertain. This point of view can be found, e.g., in [21], where the trapezoidal approximation with respect to the weighted distance

$$d_{ZL}(A,T(A)) = \left(\int_0^1 \alpha[A_L(\alpha) - T(A)_L(\alpha)]^2 d\alpha + \int_0^1 \alpha[A_U(\alpha) - T(A)_U(\alpha)]^2 d\alpha \right)^{1/2} \quad (8)$$

with increasing weighting function is applied.

However, on may also consider another weighted distances which are more appropriate in some other occasions. Let us observe that the least informative α-cut is not zero but 0.5. Actually, situation $\mu_A(x) = 1$ leads to perfect information that x surely belongs to A. If $\mu_A(x)$ is close to 1 we'll say that x rather belongs to A. And conversely, $\mu_A(x) = 0$ shows that x surely does not belong to A (and belongs to $\neg A$) which is also a perfect information. Similarly, x such that $\mu_A(x)$ is close to 0 is interpreted as a point that rather does not belong to A. However, if

$\mu_A(x) = 0.5$ we do not know whether x should be classified to A or to it's completion $\neg A$ because it belong to these sets with the same degree. Similar problems with interpretation happen if $\mu_A(x)$ is close to 0.5. Thus it is clear that degrees of membership both high (close to 1) and low (close to 0) are much more informative than those close to 0.5. Hence, if we try to incorporate this obvious conclusion into practice we have to consider not increasing weighted distance (8) but so-called bi-symmetrical weighted distance suggested in [15]. More precisely, the authors of [15] considered following two bi-symmetrical weighted functions

$$\lambda_1(\alpha) = \begin{cases} 1-2\alpha & \text{if} \quad \alpha \in [0, \frac{1}{2}], \\ 2\alpha - 1 & \text{if} \quad \alpha \in [\frac{1}{2}, 1], \end{cases} \tag{9}$$

and

$$\lambda_2(\alpha) = \begin{cases} 1 & \text{if} \quad \alpha \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1], \\ 0 & \text{if} \quad \alpha \in (\frac{1}{4}, \frac{3}{4}). \end{cases} \tag{10}$$

Function (9) is in some sense a bi-symmetrical counterpart of the increasing weighted function such as applied in [21]. The second one, contrary to previous continuous weighted function is a noncontinuous one which appreciates only elements with high or low degree of membership and does not take into account the other. In some sense (10) corresponds to Pedrycz's viewpoint expressed in his shadowed sets (see [17]) where we consider only these points which rather belong to a set under study or those that rather do not belong to it. The other elements with intermediate membership degree form the so-called shadow.

Having such bi-symmetrical weighted functions we may define a bi-symmetrical weighted distance between A and B. More precisely, for two arbitrary fuzzy numbers A and B with α-cuts $[A_L(\alpha), A_U(\alpha)]$ and $[B_L(\alpha), B_U(\alpha)]$, respectively, the quantity

$$d_\lambda(A,B)) = \left(\int_0^1 \lambda(\alpha)[A_L(\alpha) - B_L(\alpha)]^2 d\alpha + \int_0^1 \lambda(\alpha)[A_U(\alpha) - B_U(\alpha)]^2 d\alpha \right)^{1/2}, \tag{11}$$

where $\lambda \in \{\lambda_1, \lambda_2\}$ is a bi-symmetrical weighted function is called the bi-symmetrical weighted distance between A and B based on λ (see [15]).

Then, substituting B in (11) by a trapezoidal fuzzy number $T(A)$ we may try to find the nearest trapezoidal approximation to given fuzzy number A. Such problem for bi-symmetrical weighted functions λ_1 and λ_2 was solved in [15] where the authors considered a trapezoidal approximation that preserves the expected interval.

One can, of course, propose many other bi-symmetrical weighted functions having properties similar to λ_1 and λ_2. Thus it would be interesting to specify a general definition of the bi-symmetrical weighted function. Then for the defined family of bi-symmetrical weighted functions we will consider trapezoidal approximation based on the bi-symmetrical weighted distance obtained for any representant of that family.

IV. Main result

Let us adopt a following definition

Definition 1
Any function $\lambda : [0,1] \to [0,1]$ such that
(a) $\lambda(\frac{1}{2} - \alpha) = \lambda(\frac{1}{2} + \alpha)$ *for all* $\alpha \in [0, \frac{1}{2}]$,
(b) $\lambda(\frac{1}{2}) \le \lambda(\alpha)$ *for any* $\alpha \in [0,1]$,
is called a bi-symmetrical weighted function.

Thus each bi-symmetrical weighted function λ is symmetrical around $\frac{1}{2}$ and λ reaches its minimum in $\frac{1}{2}$. Further on we will examine a reach subclass of all bi-symmetrical weighted function, called regular bi-symmetrical weighted functions.

Definition 2
A bi-symmetrical weighted function λ is called regular if
(a) $\lambda(\frac{1}{2}) = 0$,
(b) $\lambda(0) = \lambda(1) = 1$,
(c) $\int_0^1 \lambda(\alpha) d\alpha = \frac{1}{2}$.

A family of all regular bi-symmetrical weighted functions will be denoted by $\mathbb{BSWF}$.

Let us go back to the trapezoidal approximation operators $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$ which produce a trapezoidal fuzzy number $T(A)$ that is the closest to given original fuzzy number A among all trapezoidal fuzzy numbers having identical expected interval as the original one, i.e. satisfying (7). However, now we will look for the operators which minimize the bi-symmetrical weighted distance (11) based on bi-symmetrical function $\lambda \in \mathbb{BSWF}$.

Since α-cuts of $T(A)$ are of the form $[t_1 + (t_2 - t_1)\,\alpha, t_4 - (t_4 - t_3)\alpha]$, thus substituting it into (11) and (7) our problem might be expressed as follows:

Problem 3
For any $\lambda \in \mathbb{BSWF}$ find $t_1, t_2, t_3, t_4 \in \mathbb{R}$ which minimize

$$d(A, T(A)) = \left(\int_0^1 \lambda(\alpha) \left[A_L(\alpha) - (t_1 + (t_2 - t_1)\,\alpha)\right]^2 d\alpha + \int_0^1 \lambda(\alpha) \left[A_U(\alpha) - (t_4 - (t_4 - t_3)\alpha)\right]^2 d\alpha \right)^{1/2} \tag{12}$$

with respect to conditions

$$\frac{t_1 + t_2}{2} = \int_0^1 A_L(\alpha) d\alpha, \tag{13}$$

$$\frac{t_3 + t_4}{2} = \int_0^1 A_U(\alpha) d\alpha \tag{14}$$

$$t_1 \le t_2 \le t_3 \le t_4. \tag{15}$$

Solving the given problem we have noticed that some useful notation should be introduced to obtain the final result in a nice form with clear interpretation. Firstly let us notice that by Def. 2 the centroid of a bi-symmetrical weighted function λ is $\frac{1}{2}$. Therefore, the dispersion of the bi-symmetrical weighted function λ is given by

$$\eta = \int_0^1 (\alpha - \frac{1}{2})^2 \lambda(\alpha)\, d\alpha. \tag{16}$$

We also introduce another two parameters characterizing the dispersion of the left side and of the right side of given fuzzy number A with respect to considered bi-symmetrical weighted function λ.

Definition 4
The left (lower) spread of a fuzzy number A with respect to considered bi-symmetrical weighted function λ is a number $LSP_\lambda(A)$ given by

$$LSP_\lambda(A) = \frac{1}{2\eta}\int_0^1 (\alpha - \frac{1}{2})\lambda(\alpha) A_L(\alpha)\, d\alpha, \tag{17}$$

while the right (upper) spread of a fuzzy number A with respect to considered bi-symmetrical weighted function λ is a following number $USP_\lambda(A)$

$$USP_\lambda(A) = \frac{1}{2\eta}\int_0^1 (\frac{1}{2} - \alpha)\lambda(\alpha) A_U(\alpha)\, d\alpha. \tag{18}$$

It can be shown that $LSP_\lambda(A) \geq 0$ and $USP_\lambda(A) \geq 0$.

Let us also denote the total spread of a given fuzzy number A with respect to considered bi-symmetrical weighted function λ by $TSP_\lambda(A)$, i.e.

$$TSP_\lambda(A) = LSP_\lambda(A) + USP_\lambda(A). \tag{19}$$

The difference between the right and the left spread of a fuzzy number will be denoted by $\Delta SP_\lambda(A)$, i.e.

$$\Delta SP_\lambda(A) = USP_\lambda(A) - LSP_\lambda(A). \tag{20}$$

It is easily seen that as $TSP_\lambda(A)$ is always nonnegative, while $\Delta SP_\lambda(A)$ might be positive or negative as A is more asymmetrical to the right or to the left.

We can now formulate our main result, i.e. the solution of Problem 3.

Theorem 5
For any regular bi-symmetrical weighted function $\lambda \in \mathbb{BSWF}$ the nearest trapezoidal approximation operator preserving expected interval with respect to distance (12) based on λ is such operator $T : \mathbb{F}(\mathbb{R}) \to \mathbb{F}^T(\mathbb{R})$, that for any fuzzy number A with α-cuts $[A_L(\alpha), A_U(\alpha)]$ assigns the trapezoidal fuzzy number $T(A) = T(t_1, t_2, t_3, t_4)$, where

(a) *if $w(A) \geq TSP_\lambda(A)$ then*

$$t_1 = EI_L(A) - LSP_\lambda(A), \tag{21}$$
$$t_2 = EI_L(A) + LSP_\lambda(A), \tag{22}$$
$$t_3 = EI_U(A) - USP_\lambda(A), \tag{23}$$
$$t_4 = EI_U(A) + USP_\lambda(A); \tag{24}$$

(b) *if $|\Delta SP_\lambda(A)| \leq w(A) < TSP_\lambda(A)$ then*

$$t_1 = EI_L(A) - \frac{1}{2}w(A) + \frac{1}{2}\Delta SP_\lambda(A), \tag{25}$$
$$t_2 = t_3 = EV(A) - \frac{1}{2}\Delta SP_\lambda(A), \tag{26}$$
$$t_4 = EI_U(A) + \frac{1}{2}w(A) + \frac{1}{2}\Delta SP_\lambda(A); \tag{27}$$

(c) *if $w(A) < \Delta SP_\lambda(A)$ then*

$$t_1 = t_2 = t_3 = EI_L(A), \tag{28}$$
$$t_4 = 2EI_U(A) - EI_L(A); \tag{29}$$

(d) *if $\Delta SP_\lambda(A) < 0$ and $w(A) < |\Delta SP_\lambda(A)|$ then*

$$t_1 = 2EI_L(A) - EI_U(A), \tag{30}$$
$$t_2 = t_3 = t_4 = EI_U(A). \tag{31}$$

To prove this theorem the minimization problem under given constraints should be performed and hence the Karush-Kuhn-Tucker theorem would be useful here. However, because of lack of space the proof will be omitted here.

Actually, we have received not a single operator but four different operators providing the nearest trapezoidal fuzzy number that preserves the expected value of the original fuzzy number, where T_1 leads to a regular trapezoidal fuzzy number, T_2 stands for the operator that leads to a triangular fuzzy number with two sides, while T_3 and T_4 produce triangular fuzzy numbers with the right or the left side only, respectively.

In other words, we approximate a fuzzy number A by the trapezoidal approximation operator T_1 provided the total dispersion of the given fuzzy number with respect to the considered bi-symmetrical weighted function measured by the sum of the lower and upper spread is large enough. Otherwise, we will approximate A by a triangular fuzzy number. However, for less dispersed fuzzy numbers we have three possible situations: to approximate a fuzzy number A we apply operator T_2 provided the asymmetry of A is not too big (i.e. there is no big difference between the lower and upper spread). If A reveals high right asymmetry (i.e. the right spread is significantly larger than the lower spread) it would be approximated by a triangular fuzzy number with the right side only, produced by operator T_3. Otherwise, a fuzzy number with high left asymmetry would be approximated by a triangular fuzzy number with the left side only, produced by operator T_4.

To sum up we get a following algorithm for computing the nearest trapezoidal approximation with the bi-symmetrical weighted function and preserving the expected interval:

Algorithm 6

For any fuzzy number A

Step 1. *If $w(A) \geq TSP_\lambda(A)$ then apply operator T_1 given by (21)-(24), else*

Step 2. *if $|\Delta SP_\lambda(A)| \leq w(A)$ then apply operator T_2 given by (25)-(27), else*

Step 3. *if $w(A) < \Delta SP_\lambda(A)$ then apply operator T_3 given by (28)-(29), else*

Step 4. *apply operator T_4 given by (30)-(31).*

It is interesting that in the trapezoidal approximation with the bi-symmetrical weighted function we have obtained four possible solutions like in the problem with non-weighted distance considered in [3] or [11]. Moreover, operators T_3 and T_4 are identical as in [3] or [11]. It means that for very asymmetrical fuzzy numbers its nearest trapezoidal approximation preserving the expected interval remains independent whether we use weighted or non-weighted distance.

Looking for parallels with the problem with non-weighted distance considered in [3] or [11] we can also notice that the family $\mathbb{F}(\mathbb{R})$ of all fuzzy numbers can be considered as a union of four subfamilies $\mathbb{F}_i(\mathbb{R})$ corresponding to different approximation operators to be used. We may say that a fuzzy number A belongs to subfamily $\mathbb{F}_i(\mathbb{R})$ if and only if T_i $(i = 1, \ldots, 4)$ is an appropriate operator that should be used for getting a proper trapezoidal approximation. One may notice that subfamilies $\mathbb{F}_1(\mathbb{R}), \ldots, \mathbb{F}_4(\mathbb{R})$ form a partition of a family of all fuzzy numbers $\mathbb{F}(\mathbb{R})$, i.e.

$$\mathbb{F}_1(\mathbb{R}) \cup \ldots \cup \mathbb{F}_4(\mathbb{R}) = \mathbb{F}(\mathbb{R}) \tag{32}$$

and

$$\mathbb{F}_i(\mathbb{R}) \cap \mathbb{F}_j(\mathbb{R}) = \emptyset \tag{33}$$

for $i \neq j$.

V. Conclusion

In the present paper we have considered the problem of the trapezoidal approximation of fuzzy numbers for bi-symmetrical weighted functions. This contribution is a generalization of the paper [15] where only two particular bi-symmetrical weighted functions were discussed.

One may ask whether suggested trapezoidal approximation operators fulfil requirements specified in [13]. It is clear that some of them hold. But the more detailed study would be the topic of our further research.

References

[1] S. Abbasbandy, B. Asady. *The nearest approximation of a fuzzy quantity in parametric form*. Applied Mathematics and Computation, 172:624-632, 2006.

[2] S. Abbasbandy, M. Amirfakhrian. *The nearest trapezoidal form of a generalized LR fuzzy number*. International Journal of Approximate Reasoning, 43:166-178, 2006.

[3] A. Ban. *Approximation of fuzzy numbers by trapezoidal fuzzy numbers preserving the expected interval*. Fuzzy Sets and Systems 159:1327-1344, 2008.

[4] S. Chanas. *On the interval approximation of a fuzzy number*. Fuzzy Sets and Systems, 122:353–356, 2001.

[5] M. Delgado, M.A. Vila, W. Voxman. *On a canonical representation of a fuzzy number*. Fuzzy Sets and Systems, 93:125–135, 1998.

[6] D. Dubois, H. Prade. *Operations on fuzzy numbers*. Int. J. Syst. Sci., 9:613–626, 1978.

[7] D. Dubois, H. Prade. *The mean value of a fuzzy number*. Fuzzy Sets and Systems, 24:279–300, 1987.

[8] P. Grzegorzewski. *Metrics and orders in space of fuzzy numbers*. Fuzzy Sets and Systems, 97:83–94, 1998.

[9] P. Grzegorzewski. *Nearest interval approximation of a fuzzy number*. Fuzzy Sets and Systems, 130:321–330, 2002.

[10] P. Grzegorzewski. *Approximation of a fuzzy number preserving entropy-like nonspecifity*. Operations Research and Decisions, 4:49–59, 2003.

[11] P. Grzegorzewski. *Trapezoidal approximations of fuzzy numbers preserving the expected interval - algorithms and properties*. Fuzzy Sets and Systems, 159:1354-1364, 2008.

[12] P. Grzegorzewski. *New algorithms for trapezoidal approximation of fuzzy numbers preserving the expected interval*. In: Proceedings of the Twelfth International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU'08, L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (Eds.), Spain, Torremolinos (Málaga), June 22-27, pp. 117-123, 2008.

[13] P. Grzegorzewski, E. Mrówka. *Trapezoidal approximations of fuzzy numbers*. Fuzzy Sets and Systems, 153:115–135, 2005.

[14] P. Grzegorzewski, E. Mrówka. *Trapezoidal approximations of fuzzy numbers - revisited*. Fuzzy Sets and Systems, 158:757–768, 2007.

[15] P. Grzegorzewski, K. Pasternak-Winiarska. *Weighted trapezoidal approximations of fuzzy numbers*. In: Proceedings of IFSA World congress and Eusflat Conference IFSA/Eusflat 2009, Lisbon, Portugal, 20-24 July, 2009, pp. 1531-1534.

[16] S. Heilpern. *The expected value of a fuzzy number*. Fuzzy Sets and Systems, 47:81–86, 1992.

[17] W. Pedrycz. *Shadowed sets: representing and processing fuzzy sets*. IEEE Trans. Syst. Man. Cybern. B, 2:103–109, 1998.

[18] C.T. Yeh. *A note on trapezoidal approximations of fuzzy numbers*. Fuzzy Sets and Systems, 158:747–754, 2007.

[19] C.T. Yeh. *Trapezoidal and triangular approximations preserving the expected interval*. Fuzzy Sets and Systems, 159:1345-1353, 2008.

[20] C.T. Yeh. *On improving trapezoidal and triangular approximations of fuzzy numbers*. International Journal of Approximate Reasoning, 48:297-313, 2008.

[21] W. Zeng, H. Li. *Weighted triangular approximation of fuzzy numbers*. International Journal of Approximate Reasoning, 46:137-150, 2007.