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Some parametric forms for LR fuzzy numbers and LR fuzzy arithmetic

Laerte Sorini and Luciano Stefanini
University of Urbino "Carlo Bo" , Italy
laerte@uniurb.it (L. Sorini), lucste@uniurb.it (L. Stefanini)

Abstract

In this paper we show that the models for parametric representation of fuzzy numbers in the level-cuts setting can be used to model LR fuzzy numbers and LR fuzzy arithmetic. This extends the family of LR fuzzy numbers to a sequence of finite-dimensional subspaces, approximating the space of fuzzy numbers with increasing goodness. The basic arithmetic with parametric LR fuzzy numbers is illustrated in an algorithmic framework.

1 Introduction

The arithmetic with fuzzy numbers, according to Zadeh's Extension Principle, can be performed using two general settings:

a. the well known LR fuzzy numbers, for which the operations are performed by calculating the membership function $x \to \mu(x)$ of the result from the membership functions of the operands ([2], [3]).

b. the interval-based operations performed using the $\alpha-cut$ representation $\alpha \to u_\alpha^-, \alpha \to u_\alpha^+$ of the operands for each cut $[u_\alpha^-, u_\alpha^+]$. Recently, this approach has been used to develop the LU parametric representation and arithmetic ([5], [6] and extensively described in [7]).

In this paper, we show that also for LR fuzzy numbers, parametric representations can be used to model the shapes of the membership functions and to obtain operators for the fuzzy arithmetic operations. We suggest a parametrization for the LR fuzzy numbers, similar to the work done for the LU parametrization in [6] and [7].

In terms of the parameters representing the LR fuzzy numbers, it is possible to define the operators for the fuzzy arithmetic in such a way that the errors of the approximations can be reduced to any small tolerance (by increasing the number of parameters); in fact, within the space of differentiable fuzzy numbers, the approximations form a dense subspace.

An advantage of the proposed parametrization is that the arithmetic operators are valid also for fuzzy numbers having different shapes (e.g. mixing linear, quadratic, gaussian, logistic, etc.).

The parametric LR fuzzy numbers can approximate general fuzzy numbers with any desired precision; we illustrate the approximation of fuzzy numbers (e.g. polynomial, Gaussian, logistic) by the proposed parametrization and we suggest a general least-squares method to estimate the parameters.

2 LR parametric fuzzy numbers

LR-fuzzy numbers (intervals) are fuzzy sets defined over the real numbers $\mathbb{R}$, having membership functions $(x, \mu_u(x))$ for each $x \in \mathbb{R}$, in the form

$$\mu_u(x) = \begin{cases} L(\frac{b-x}{b-a}) & \text{if } x \in [a,b] \\ 1 & \text{if } x \in [b,c] \\ R(\frac{x-c}{d-c}) & \text{if } x \in [c,d] \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

where $L, R : [0,1] \longrightarrow [0,1]$ are non-increasing with $R(0) = L(0) = 1$ and $R(1) = L(1) = 0$.

The $\alpha - cuts$ are defined by $[u]_\alpha = \{x | x \in \mathbb{X}, \mu_u(x) \geq \alpha\}$ with $[u]_0 = cl\{x | x \in \mathbb{X}, \mu_u(x) > 0\}$. We will denote them by

$$\begin{aligned} [u]_\alpha &= [u_\alpha^-, u_\alpha^+] \text{ where} \\ u_\alpha^- &= b - (b-a)L^{-1}(\alpha) \\ u_\alpha^+ &= c + (d-c)R^{-1}(\alpha) \end{aligned} \tag{2}$$

It is well known that the two functions $\alpha \to u_\alpha^-$ (the Lower branch) and $\alpha \to u_\alpha^+$ (the Upper branch) are monotonic (respectively increasing and decreasing) functions for all $\alpha \in [0,1]$. The parametric representations for LR fuzzy numbers, use monotonic interpolation by shape functions $p : [0,1] \longrightarrow [0,1]$ such that $p(0) = 0$ and $p(1) = 1$ with $p(t)$ differentiable and increasing on $[0,1]$; with parameters $\beta_i \geq 0, i = 0,1$ we satisfy conditions

$$\begin{aligned} p(0) &= 0, \ p(1) = 1 \\ p'(0) &= \beta_0, \ p'(1) = \beta_1. \end{aligned}$$

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Two examples are the following (see [6], where the properties and advantages are described):

- the (2,2)-rational monotonic spline

$$p_{R2}(t;\beta_0,\beta_1)=\frac{t^2+\beta_0 t(1-t)}{1+(\beta_0+\beta_1-2)t(1-t)}; \qquad (3)$$

- the mixed polynomial-exponential spline ($k\geq 2$)

$$p_k(t;\beta_0,\beta_1)= \qquad (4)$$
$$\frac{kt^{k-1}-(k-1)t^k+\beta_0-\beta_0(1-t)^\gamma+\beta_1 t^\gamma}{\gamma}$$

where $\gamma=1+\beta_0+\beta_1$.

Function p in (3) or (4) is increasing on [0,1] and is used as model for functions L and R.

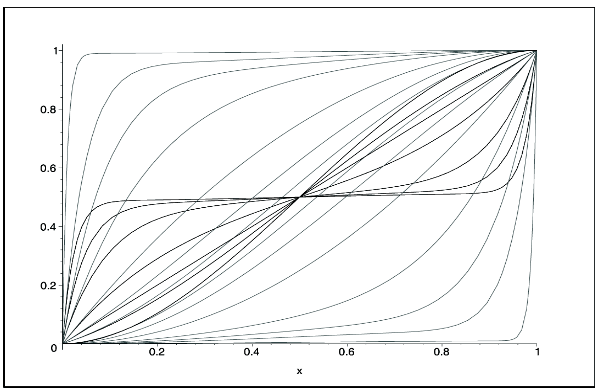

Figure 1: Standard functions (4) with different β_0,β_1; by changing the parameters, we can obtain a large family of monotonic shape functions of very different forms.

If $a\leq b\leq c\leq d$ and $\beta_{0,L},\beta_{1,L}\geq 0$, $\beta_{0,R},\beta_{1,R}\geq 0$ are given, an LR-parametric fuzzy number has membership

$$\mu_u(x)=\begin{cases} p(\frac{x-a}{b-a};\beta_{0,L},\beta_{1,L}) & \text{if } x\in[a,b]\\ 1 & \text{if } x\in[b,c]\\ p(\frac{d-x}{d-c};\beta_{0,R},\beta_{1,R}) & \text{if } x\in[c,d]\\ 0 & \text{otherwise}\end{cases}.$$

We denote a,b,c,d as $a=u_{0,L}$, $b=u_{1,L}$, $c=u_{1,R}$, $d=u_{0,R}$ so that eight parameters define u:

$$u_{LR}=(u_{0,L},\beta_{0,L},u_{0,R},\beta_{0,R};u_{1,L},\beta_{1,L},u_{1,R},\beta_{1,R}) \qquad (5)$$

provided that $u_{0,L}\leq u_{1,L}\leq u_{1,R}\leq u_{0,R}$ and $\beta_{0,L},\beta_{1,L}\geq 0$, $\beta_{0,R},\beta_{1,R}\geq 0$.

The LU representation is obtained if the model functions $p(t;\beta_0,\beta_1)$ are used to model the Lower and the Upper branches of the $\alpha-cuts$, i.e.

$$u_{LU} = (u_0^-,\beta_0^-,u_0^+,\beta_0^+;\ u_1^-,\beta_1^-,u_1^+,\beta_1^+) \qquad (6)$$

and

$$u_\alpha^- = u_0^-+(u_1^- -u_0^-)p\left(\alpha;\beta_0^-,\beta_1^-\right)$$
$$u_\alpha^+ = u_0^+ +(u_1^+ -u_0^+)p\left(\alpha;\beta_0^+,\beta_1^+\right)$$

We denote by $\mathbb{F}^{LR}$ and by $\mathbb{F}^{LU}$ the families of fuzzy numebrs in LR and LU parametric forms (5) and (6), respectively. It will be useful to use a different notation for the nonnegative parameters $\beta_{0,L},\beta_{0,R}$, $\beta_{1,L},\beta_{1,R}$ in (5) and β_0^-,β_0^+, β_1^-,β_1^+ in (6), i.e. for two fuzzy numbers u,v we denote by $\delta u_{i,L},\delta u_{i,R}$ and $\delta v_{i,L},\delta v_{i,R}$ the values of $\beta_{i,L},\beta_{i,R}$ (i=1,2) for u and v respectively and denote the LR parametrization as follows:

$$\begin{aligned} u_{LR} &= (u_{0,L},\delta u_{0,L},u_{0,R},\delta u_{0,R}; &(7)\\ &\quad u_{1,L},\delta u_{1,L},u_{1,R},\delta u_{1,R}),\\ v_{LR} &= (v_{0,L},\delta v_{0,L},v_{0,R},\delta v_{0,R}; &(8)\\ &\quad v_{1,L},\delta v_{1,L},v_{1,R},\delta v_{1,R}).\end{aligned}$$

Denote by $\mathbb{F}_1^{LR}$ the set of LR-fuzzy numbers defined by (5). In its simpler form, the family of fuzzy numbers $\mathbb{F}_1^{LR}$, which include triangular and trapezoidal fuzzy numbers if all $\beta_{i,L}=\beta_{i,R}=1$, are characterized by eight parameters characterizing the core $[u_{1,L},u_{1,R}]$ (corresponding to $\alpha=\alpha_1$) and the support $[u_{0,L},u_{0,R}]$ (corresponding to $\alpha=\alpha_0$); the four parameters $\delta u_{0,L},\delta u_{0,R}$ and $\delta u_{1,L},\delta u_{1,R}$ are used for the first derivatives of $\mu_u(x)$ at the extremal point of the two intervals above.

More generally, we can design differentiable parametric shape functions $L()$ and $R()$ by fixing $N+1$ distinct level sets $0=\alpha_0<\alpha_1<...<\alpha_N=1$ and assigning, for each α_i ($i=0,1,...,N$), the four parameters needed for each level.

In this general case, we need the values $u_{i,L},u_{i,R}$ but, instead of the parameters $\delta u_{i,L},\delta u_{i,R}$, it is convenient to give directly the first derivatives of $\mu_u(x)$ at $u_{i,L}$, $u_{i,R}$ and determine the parameters $\delta u_{i,L},\delta u_{i,R}\geq 0$ in the monotonic functions like (3) and (4); the membership function is then defined piecewise on each subinterval $[u_{i-1,L},u_{i,L}]$ and $[u_{i,R},u_{i-1,R}]$, $i=0,1,...,N$.

For $i=0,1,...,N$, denote directly by $\delta u_{i,L}\geq 0$ and by $\delta u_{i,R}\leq 0$ the first derivatives of $\mu_u(x)$ at the points $x=u_{i,L}$ and $x=u_{i,R}$, respectively and suppose they are given. The membership function $\mu_u(x)$ is then obtained by the following simple procedure:

Assign first $\mu_u(x)=0$ for all x;
for all $x\in[u_{N,L},u_{N,R}]$ set $\mu_u(x)=1$;
for $i=1,2,...,N$
  for $x\in[u_{i-1,L},u_{i,L}]$ set
    $\beta_0=\frac{u_{i,L}-u_{i-1,L}}{\alpha_i-\alpha_{i-1}}\delta u_{i-1,L}$
    $\beta_1=\frac{u_{i,L}-u_{i-1,L}}{\alpha_i-\alpha_{i-1}}\delta u_{i,L}$
    $\mu_u(x)=\alpha_{i-1}+(\alpha_i-\alpha_{i-1})p(\frac{x-u_{i-1,L}}{u_{i,L}-u_{i-1,L}};\beta_0,\beta_1)$
  end
  for $x\in[u_{i,R},u_{i-1,R}]$ set
    $\beta_0=\frac{u_{i,R}-u_{i-1,R}}{\alpha_i-\alpha_{i-1}}\delta u_{i-1,R}$
    $\beta_1=\frac{u_{i,R}-u_{i-1,R}}{\alpha_i-\alpha_{i-1}}\delta u_{i,R}$

$$\mu_u(x) = \alpha_{i-1} + (\alpha_i - \alpha_{i-1}) p(\frac{x-u_{i-1,R}}{u_{i,R}-u_{i-1,R}}; \beta_0, \beta_1)$$

end

end

Denote by $\mathbb{F}_N^{LR}$ the fuzzy numbers obtained in the form above; as each shape function is monotonic, the left and right branches are monotonic increasing on $[u_{0,L}, u_{N,L}]$ (the left side of μ_u) and decreasing on $[u_{N,R}, u_{0,R}]$ (the right side of μ_u). The number of parameters is $4N + 4$ and they are simply constrained to be $u_{0,L} \le ... \le u_{N,L} \le u_{N,R} \le ... \le u_{0,R}$ and $\delta u_{i,L} \ge 0$ and $\delta u_{i,R} \le 0$ to ensure monotonicity. In general, we will have $u_{i-1,L} < u_{i,L}$ and $u_{i-1,R} > u_{i,R}$ but it is easy to consider the case of equality so that the graph of μ_u is a vertical line (discontinuity) at $u_{i,.}$ if $u_{i-1,.} = u_{i,.}$.

In this general form, an LR fuzzy number is represented as follows:

$$u = (\alpha_i; u_{i,L}, \delta u_{i,L}, u_{i,R}, \delta u_{i,R})_{i=0,1,...,N}. \quad (9)$$

Example: Consider a quasi-Gaussian membership function ($m \in \mathbb{R}$, $k, \sigma \in \mathbb{R}^+$; if $k \to +\infty$ the support is unbounded)

$$\mu_{qG}(x) = \begin{cases} \exp(-\frac{(x-m)^2}{2\sigma^2}) & \text{if } m - k\sigma \le x \le m + k\sigma \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

Its LR parametrization for $m = 0$, $\sigma = 2$, $k = 4$, approximated with $N = 4$(five points), is

α_i	$u_{i,L}$	$\delta u_{i,L}$	$u_{i,R}$	$\delta u_{i,R}$
0.0	−8.0	0.00033	8.0	−0.00033
0.25	−3.33022	0.20814	3.33022	−0.20814
0.5	−2.35482	0.29435	2.35482	−0.29435
0.75	−1.51705	0.28445	1.51705	−0.28445
1.0	0.0	0.0	0.0	0.0

Table 1: LR parametrization of fuzzy number (10).

Using a parametrization of the form 9 as in Table 1, we can represent a very large family of fuzzy numbers, and in particular we can approximate with good quality (by eventually increasing N) fuzzy numbers originating from different standard Left and Right shape functions; furthermore, the arithmetic operations will be based directly on the parameters appearing in 9, so whatever the fuzzy numbers are, the same rules are applied.

The generated LR fuzzy numbers form a subspace of the space of fuzzy numbers. In the case of differentiable membership functions, it is immediate to understand that the union of all parametric fuzzy numbers having the form (9) for all integer $N \ge 1$, i.e.

$$\mathcal{F}^{LR} = \bigcup_{N \ge 1} \mathbb{F}_N^{LR},$$

is dense into the space of (differentiable) fuzzy numbers. In fact, each fuzzy number (9) has the property of interpolate (exactly) the values $u_{i,L}, u_{i,R}$ ($i = 0, 1, ..., N$) and it is sufficient to refine the points α_i to obtain any desired precision. In our experience, approximations with values of N from 5 to 20 have in general a very small error, of the order of $10^{-6} - 10^{-2}$.

We can define a geometric distance $D_2(u, v)$ between fuzzy numbers $u, v \in \mathbb{F}_N^{LR}$, given by

$$D_2^{LR}(u,v) = \left(\sum_{i=0}^{N} |u_{i,L} - v_{i,L}|^2 + |u_{i,R} - v_{i,R}|^2 \right)^{1/2} + \\ + \left(\sum_{i=0}^{N} |\delta u_{i,L} - \delta v_{i,L}|^2 + |\delta u_{i,R} - \delta v_{i,R}|^2 \right)^{1/2}.$$

Remark: *If we model the LR-fuzzy numbers by a (2,2)-rational spline $p(\alpha; \beta_0, \beta_1)$ like (3) the inverse $p^{-1}(t; \beta_0, \beta_1)$ can be computed analytically as we have to solve the quadratic equation (with respect to α)*

$$\alpha^2 + \beta_0 \alpha(1-\alpha) = t[1 + (\beta_0 + \beta_1 - 2)\alpha(1-\alpha)] \text{ i.e.}$$

$$(1 + A(t))\alpha^2 - A(t)\alpha - t = 0 \text{ where}$$
$$A(t) = -\beta_0 + \beta_0 t + \beta_1 t - 2t\,;$$

if $A(t) = -1$ then the equation is linear and the solution is $\alpha = t$. If $A(t) \ne -1$, then there exist two real solutions $\alpha_1 = \frac{2\sqrt{t}}{2+2A(t)}$, $\alpha_2 = \frac{2\sqrt{t}+2A(t)}{2+2A(t)}$ and we choose the one belonging to $[0, 1]$.

3 Fuzzy arithmetic with LR parametrization

The fuzzy extension principle introduced by Zadeh is the basic tool for fuzzy calculus; it extends functions of real numbers to functions of fuzzy numbers and it allows the extension of arithmetic operations and calculus to fuzzy arguments.

3.1 Basic arithmetic operators

Let $\circ \in \{+, -, \times, /\}$ one of the four arithmetic operations and let $u, v \in \mathbb{F}_\mathbb{I}$ be given fuzzy intervals (or numbers), having $\mu_u(.)$ and $\mu_v(.)$ as membership functions and level-cuts representations $u = (u^-, u^+)$, $v = (v^-, v^+)$; the extension principle for the extension of $\circ$ defines the membership function of $w = u \circ v$ by

$$\mu_{u \circ v}(z) = \sup\{\min\{\mu_u(x), \mu_v(y)\} | z = x \circ y\}. \quad (11)$$

In terms of the $\alpha - cuts$, the four arithmetic operations and the scalar multiplication for $k \in \mathbb{R}$ are obtained by the well-known *interval arithmetic* (for all $\alpha \in [0, 1]$)

Addition/Subtraction ($w = u \pm v$):
$[u + v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$,
$[u - v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-]$,

Scalar multiplication ($w = ku$):

$[ku]_\alpha = [\min\{ku^-_\alpha, ku^+_\alpha\}, \max\{ku^-_\alpha, ku^+_\alpha\}]$,

Multiplication ($w = uv$):

$$\begin{cases} (uv)^-_\alpha = \min\{u^-_\alpha v^-_\alpha, u^-_\alpha v^+_\alpha, u^+_\alpha v^-_\alpha, u^+_\alpha v^+_\alpha\} \\ (uv)^+_\alpha = \max\{u^-_\alpha v^-_\alpha, u^-_\alpha v^+_\alpha, u^+_\alpha v^-_\alpha, u^+_\alpha v^+_\alpha\} \end{cases},$$

Division ($w = u/v$): if $0 \notin [v^-_0, v^+_0]$

$$\begin{cases} (u/v)^-_\alpha = \min\left\{\frac{u^-_\alpha}{v^-_\alpha}, \frac{u^-_\alpha}{v^+_\alpha}, \frac{u^+_\alpha}{v^-_\alpha}, \frac{u^+_\alpha}{v^+_\alpha}\right\} \\ (u/v)^+_\alpha = \max\left\{\frac{u^-_\alpha}{v^-_\alpha}, \frac{u^-_\alpha}{v^+_\alpha}, \frac{u^+_\alpha}{v^-_\alpha}, \frac{u^+_\alpha}{v^+_\alpha}\right\} \end{cases}.$$

Consider two LR-fuzzy numbers u and v ($N = 1$ for simplicity)

$$\begin{aligned} u &= (u_{0,L}, \delta u_{0,L}, u_{0,R}, \delta u_{0,R}; \quad (12) \\ &\quad u_{1,L}, \delta u_{1,L}, u_{1,R}, \delta u_{1,R}), \\ v &= (v_{0,L}, \delta v_{0,L}, v_{0,R}, \delta v_{0,R}; \quad (13) \\ &\quad v_{1,L}, \delta v_{1,L}, v_{1,R}, \delta v_{1,R}). \end{aligned}$$

Remark: *Note that, in the formulae below, u and v are not restricted to have the same $L(.)$ and $R(.)$ shape functions and changing the slopes will change the form of the membership functions. This allows arithmetic operator for approximated arithmetic involving LR fuzzy numbers of different and general shape.*

The addition is the following

$$(u+v) = \begin{pmatrix} u_{0,L} + v_{0,L}, \frac{\delta u_{0,L}\delta v_{0,L}}{\delta u_{0,L}+\delta v_{0,L}}, \\ u_{0,R} + v_{0,R}, \frac{\delta u_{0,R}\delta v_{0,R}}{\delta u_{0,R}+\delta v_{0,R}}; \\ u_{1,L} + v_{1,L}, \frac{\delta u_{1,L}\delta v_{1,L}}{\delta u_{1,L}+\delta v_{1,L}}, \\ u_{1,R} + v_{1,R}, \frac{\delta u_{1,R}\delta v_{1,R}}{\delta u_{1,R}+\delta v_{1,R}} \end{pmatrix}.$$

In all the cases, we will assume conventionally that $\frac{0}{0} = 0$; on the other hand, if $\delta u_{1,L} = \delta v_{1,L} = 0$, then $\delta u_{0,L}\delta v_{0,L} = (0^2)$. In the calculations, we "approximate" $\pm\infty$ by $\pm BIG$ where BIG is a big positive number.

Note that if the left and right shapes of u and v are the same (e.g. linear, quadratic) then the addition is exact.

The general algorithm for approximate LR addition is

Algorithm (*LR addition*) $w = u + v$
for $i = 0, 1, ..., N$
 $w_{i,L} = u_{i,L} + v_{i,L}$, $w_{i,R} = u_{i,R} + v_{i,R}$
 $\delta w_{i,L} = \frac{\delta u_{i,L}\delta v_{i,L}}{\delta u_{i,L}+\delta v_{i,L}}$, $\delta w_{i,R} = \frac{\delta u_{i,R}\delta v_{i,R}}{\delta u_{i,R}+\delta v_{i,R}}$
end

The difference $w = u - v$ is similar

Algorithm (*LR subtraction*) $w = u - v$
for $i = 0, 1, ..., N$
 $w_{i,L} = u_{i,L} - v_{i,R}$, $w_{i,R} = u_{i,R} - v_{i,L}$
 $\delta w_{i,L} = \frac{\delta u_{i,L}\delta v_{i,R}}{\delta v_{i,R}-\delta u_{i,L}}$, $\delta w_{i,R} = \frac{\delta u_{i,R}\delta v_{i,L}}{\delta v_{i,L}-\delta u_{i,R}}$
end

Also the approximate multiplication $w = uv$ can be obtained easily; in the case of two positive LR-fuzzy numbers w is given by

$$w_{LR} = \begin{pmatrix} u_{0,L}v_{0,L}, \frac{\delta u_{0,L}\delta v_{0,L}}{v_{0,L}\delta v_{0,L}+u_{0,L}\delta u_{0,L}}, \\ u_{0,R}v_{0,R}, \frac{\delta u_{0,R}\delta v_{0,R}}{v_{0,R}\delta v_{0,R}+u_{0,R}\delta u_{0,R}}; \\ u_{1,L}v_{1,L}, \frac{\delta u_{1,L}\delta v_{1,L}}{v_{1,L}\delta v_{1,L}+u_{1,L}\delta u_{1,L}}, \\ u_{1,R}v_{1,R}, \frac{\delta u_{1,R}\delta v_{1,R}}{v_{1,R}\delta v_{1,R}+u_{1,R}\delta u_{1,R}} \end{pmatrix}.$$

The general algorithm for approximate LR multiplication is

Algorithm (*LR multiplication*) $w = uv$
(eventually, use $\frac{0}{0} = 0$ and $\frac{c}{0} = \pm BIG$)
for $i = 0, 1, ..., N$
 $m_i = \min\{u_{i,L}v_{i,L}, u_{i,L}v_{i,R}, u_{i,R}v_{i,L}, u_{i,R}v_{i,R}\}$
 $M_i = \max\{u_{i,L}v_{i,L}, u_{i,L}v_{i,R}, u_{i,R}v_{i,L}, u_{i,R}v_{i,R}\}$
 $w_{i,L} = m_i$, $w_{i,R} = M_i$
 if $u_{i,L}v_{i,L} = m_i$ **then** $\delta w_{i,L} = \frac{\delta u_{i,L}\delta v_{i,L}}{v_{i,L}\delta v_{i,L}+u_{i,L}\delta u_{i,L}}$
 if $u_{i,L}v_{i,R} = m_i$ **then** $\delta w_{i,L} = \frac{\delta u_{i,L}\delta v_{i,R}}{v_{i,R}\delta v_{i,R}+u_{i,L}\delta u_{i,L}}$
 if $u_{i,R}v_{i,L} = m_i$ **then** $\delta w_{i,L} = \frac{\delta u_{i,R}\delta v_{i,L}}{v_{i,L}\delta v_{i,L}+u_{i,R}\delta u_{i,R}}$
 if $u_{i,R}v_{i,R} = m_i$ **then** $\delta w_{i,L} = \frac{\delta u_{i,R}\delta v_{i,R}}{v_{i,R}\delta v_{i,R}+u_{i,R}\delta u_{i,R}}$
 if $u_{i,L}v_{i,L} = M_i$ **then** $\delta w_{i,R} = \frac{\delta u_{i,L}\delta v_{i,L}}{v_{i,L}\delta v_{i,L}+u_{i,L}\delta u_{i,L}}$
 if $u_{i,L}v_{i,R} = M_i$ **then** $\delta w_{i,R} = \frac{\delta u_{i,L}\delta v_{i,R}}{v_{i,R}\delta v_{i,R}+u_{i,L}\delta u_{i,L}}$
 if $u_{i,R}v_{i,L} = M_i$ **then** $\delta w_{i,R} = \frac{\delta u_{i,R}\delta v_{i,L}}{v_{i,L}\delta v_{i,L}+u_{i,R}\delta u_{i,R}}$
 if $u_{i,R}v_{i,R} = M_i$ **then** $\delta w_{i,R} = \frac{\delta u_{i,R}\delta v_{i,R}}{v_{i,R}\delta v_{i,R}+u_{i,R}\delta u_{i,R}}$
end

The division is similar:

Algorithm (*LR division*) $w = u/v$, $0 \notin [v]_0$
(eventually, use $\frac{0}{0} = 0$ and $\frac{c}{0} = \pm BIG$)
for $i = 0, 1, ..., N$
 $m_i = \min\{u_{i,L}/v_{i,L}, u_{i,L}/v_{i,R}, u_{i,R}/v_{i,L}, u_{i,R}/v_{i,R}\}$
 $M_i = \max\{u_{i,L}/v_{i,L}, u_{i,L}/v_{i,R}, u_{i,R}/v_{i,L}, u_{i,R}/v_{i,R}\}$
 $w_{i,L} = m_i$, $w_{i,R} = M_i$
 if $u_{i,L}/v_{i,L} = m_i$ **then** $\delta w_{i,L} = \frac{v^2_{i,L}\delta u_{i,L}\delta v_{i,L}}{v_{i,L}\delta v_{i,L}-u_{i,L}\delta u_{i,L}}$
 if $u_{i,L}/v_{i,R} = m_i$ **then** $\delta w_{i,L} = \frac{v^2_{i,R}\delta u_{i,L}\delta v_{i,R}}{v_{i,R}\delta v_{i,R}-u_{i,L}\delta u_{i,L}}$
 if $u_{i,R}/v_{i,L} = m_i$ **then** $\delta w_{i,L} = \frac{v^2_{i,L}\delta u_{i,R}\delta v_{i,L}}{v_{i,L}\delta v_{i,L}-u_{i,R}\delta u_{i,R}}$
 if $u_{i,R}/v_{i,R} = m_i$ **then** $\delta w_{i,L} = \frac{v^2_{i,R}\delta u_{i,R}\delta v_{i,R}}{v_{i,R}\delta v_{i,R}-u_{i,R}\delta u_{i,R}}$
 if $u_{i,L}/v_{i,L} = M_i$ **then** $\delta w_{i,R} = \frac{v^2_{i,L}\delta u_{i,L}\delta v_{i,L}}{v_{i,L}\delta v_{i,L}-u_{i,L}\delta u_{i,L}}$
 if $u_{i,L}/v_{i,R} = M_i$ **then** $\delta w_{i,R} = \frac{v^2_{i,R}\delta u_{i,L}\delta v_{i,R}}{v_{i,R}\delta v_{i,R}-u_{i,L}\delta u_{i,L}}$
 if $u_{i,R}/v_{i,L} = M_i$ **then** $\delta w_{i,R} = \frac{v^2_{i,L}\delta u_{i,R}\delta v_{i,L}}{v_{i,L}\delta v_{i,L}-u_{i,R}\delta u_{i,R}}$
 if $u_{i,R}/v_{i,R} = M_i$ **then** $\delta w_{i,R} = \frac{v^2_{i,R}\delta u_{i,R}\delta v_{i,R}}{v_{i,R}\delta v_{i,R}-u_{i,R}\delta u_{i,R}}$
end

The operations above produce exact values at the nodes $\alpha = \alpha_i$, $i = 0, 1, ..., N$, and have very small global errors for all $\alpha \in [0,1]$ (if N is sufficiently high, of the order of 5 to 10). Further, it is easy to control the error by using a sufficiently fine $\alpha-$decomposition and the results have shown that both the rational (3) and the mixed (4) models perform well.

Some parametric membership functions in the LR framework are present in many applications and the use of nonlinear shapes in increasing. Usually, one defines a given family, e.g. linear, quadratic, sigmoid, quasi gaussian, and the operations are performed within the same family. Our proposed parametrization allows an extended set of flexible fuzzy numbers and is able to approximate all other forms with acceptable small errors with the additional advantage of producing good approximations to the results of the arithmetic operations even between LR fuzzy numbers having very different original shapes.

3.2 Computation of fuzzy-valued functions

For a continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, let $v = f(u_1, u_2, ..., u_n)$ denote its fuzzy extension, based on the application of the Zadeh's Extension Principle (*EP* for short). It is well known that the $\alpha - cuts$ $[v_\alpha^-, v_\alpha^+]$ of v are obtained by solving the following box-constrained global optimization problems ($\alpha \in [0,1]$)

$$v_\alpha^- = \min\left\{f(x_1,...,x_n)|x_k \in [u_k]_\alpha,\ k=1,2,...,n\right\} \tag{14}$$

$$v_\alpha^+ = \max\left\{f(x_1,...,x_n)|x_k \in [u_k]_\alpha,\ k=1,2,...,n\right\} \tag{15}$$

where $[u_k]_\alpha = [u_{k,\alpha}^-, u_{k,\alpha}^+]$, $k = 1,2,...,n$, are the $\alpha-cuts$ of u_k.

The lower and upper values v_α^- and v_α^+ of v define equivalently (as f is assumed to be continuous) the image of the cartesian product $\overset{n}{\underset{k=1}{\times}}[u_k]_\alpha$ via f, i.e. $[v_\alpha^-, v_\alpha^+] = f([u_1]_\alpha, ..., [u_n]_\alpha)$.

Except for simple elementary cases for which the optimization problems above can be solved analytically, the direct application of (14) and (15) is difficult and computationally expensive (see [7]).

At least if f is differentiable, the advantages of the LR representation appear to be quite interesting, based on the fact that a small number of α points is in general sufficient to obtain good approximations (this is the essential gain in using the slopes to model fuzzy numbers), so reducing the number of constrained *min* (14) and *max* (15) problems to be solved directly. On the other hand, finding computationally efficient extension solvers is still an open research field in fuzzy calculations.

We now give the details of the fuzzy extension of general differentiable functions of only one variable, by the LR representation. The case of multidimensional differentiable functions can be approached in a similar way, by considering the partial derivatives of $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and the chain rules for the composition of multidimensional functions.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$; its *(EP)*-extension $v = f(u)$ to a fuzzy argument $u = (u^-, u^+)$ has $\alpha - cuts$

$$[v]_\alpha = \left[\min\left\{f(x) \mid x \in [u]_\alpha\right\}, \max\left\{f(x) \mid x \in [u]_\alpha\right\}\right]. \tag{16}$$

If f is monotonic increasing we obtain $[v]_\alpha = [f(u_\alpha^-), f(u_\alpha^+)]$ while, if f is monotonic decreasing, $[v]_\alpha = [f(u_\alpha^+), f(u_\alpha^-)]$; for simplicity, in the monotonic case we assume that the derivative of f is not null over the support of u, but it is possible to design the algorithm also in the case where, for some α, $f'(u_\alpha^+) = 0$ or $f'(u_\alpha^-) = 0$.

In the non monotonic (differentiable) case, we have to solve the optimization problems in (16) for each $\alpha = \alpha_i$, $i = 0, 1, ..., N$, i.e.

$$(EP_i)\text{: } \begin{cases} v_{i,L} = \min\left\{f(x)|x \in [u_{i,L}, u_{i,R}]\right\} \\ v_{i,R} = \max\left\{f(x)|x \in [u_{i,L}, u_{i,R}]\right\}. \end{cases}$$

The $\min$ (or the $\max$) can occur either at a point which is coincident with one of the extremal values of $[u_{i,L}, u_{i,R}]$ or at a point which is internal; in the last case, the derivative of f is null and $\delta v_{i,L} = +\infty$ (or $\delta v_{i,R} = -\infty$); in this case, we use $\pm BIG$ where BIG is a big positive number.

The minimizations and maximizations appearing in the algorithm can be performed either analitically, or by exact or approximated algorithmes, depending on the problem at hand. In this presentation we have no space for the details; the interested reader can consult [7].

Algorithm: (1-dim non monotonic LR extension)

Let $u = (\alpha_i; u_{i,L}, \delta u_{i,L}, u_{i,R}, \delta u_{i,R})_{i=0,1,...,N}$ be given and $f : [u_{0,L}, u_{0,R}] \rightarrow \mathbb{R}$ be differentiable and monotonic; calculate $v = f(u)$.

for $i = 0, 1, ..., N$
  solve $\min\left\{f(x)|x \in [u_{i,L}, u_{i,R}]\right\}$
  let $\widehat{x}_i = \arg\min\left\{f(x)|x \in [u_{i,L}, u_{i,R}]\right\}$
  set $v_{i,L} = f(\widehat{x}_i),\ \delta v_{i,L} = BIG$
  if $\widehat{x}_i = u_{i,L}$ **then** $v_{i,L} = f(u_{i,L}),\ \delta v_{i,L} = \frac{\delta u_{i,L}}{f'(u_{i,L})}$
  if $\widehat{x}_i = u_{i,R}$ **then** $v_{i,L} = f(u_{i,R}), \delta v_{i,L} = \frac{\delta u_{i,R}}{f'(u_{i,R})}$
  solve $\max\left\{f(x)|x \in [u_{i,L}, u_{i,R}]\right\}$
  let $\widehat{x}_i = \arg\max\left\{f(x)|x \in [u_{i,L}, u_{i,R}]\right\}$
  set $v_{i,R} = f(\widehat{x}_i),\ \delta v_{i,R} = -BIG$
  if $\widehat{x}_i = u_{i,L}$ **then** $v_{i.R} = f(u_{i,L}), \delta v_{i,R} = \frac{\delta u_{i,L}}{f'(u_{i,L})}$
  if $\widehat{x}_i = u_{i,R}$ **then** $v_{i,R} = f(u_{i,R}), \delta v_{i,R} = \frac{\delta u_{i,R}}{f'(u_{i,R})}$
end

As an example of unidimensional non-monotonic function, consider the simple square function $y = x^2$. Its fuzzy extension to u can be obtained as follows:

Example: *Calculation of fuzzy* $v = u^2$.

for $i = 0, 1, ..., N$
  if $u_{i,R} \leq 0$ **then**

$v_{i,L} = (u_{i,R})^2, v_{i,R} = (u_{i,L})^2$
$\delta v_{i,L} = 2u_{i,R}\delta u_{i,R}, \delta v_{i,R} = 2u_{i,L}\delta u_{i,L}$
elseif $u_{i,L} \geq 0$ **then**
$v_{i,L} = \left(u_i^-\right)^2, v_{i,R} = \left(u_i^+\right)^2$
$\delta v_{i,L} = 2u_{i,L}\delta u_{i,L}, \delta v_{i,R} = 2u_{i,R}\delta u_{i,R}$
else
$v_{i,L} = 0,\ \delta v_{i,L} = BIG$
if $abs(u_{i,L}) \geq abs(u_{i,R})$
then $v_{i,R} = (u_{i,L})^2$, $\delta v_{i,R} = 2u_{i,L}\delta u_{i,L}$
else $v_{i,R} = (u_{i,R})^2$, $\delta v_{i,R} = 2u_{i,R}\delta u_{i,R}$
endif
endif
end

3.3 Approximation of fuzzy numbers by LR fuzzy numbers

We show in this section how the illustrated setting is useful to solve two specific types of approximations of a given fuzzy number A for which we know the support $[a, d]$ and the core $[b, c]$. The recent literature has presented various approaches and solutions (see. e.g. [1] and [4]).

If, for example, we look for the least squares best approximation of the membership, under the condition that the support and the core of the approximating fuzzy number u are exactly the same of A, in other words, $u_{0,L} = a$, $u_{0,R} = d$, $u_{1,L} = b$ and $u_{1,R} = c$, we have to estimate the shape-parameters $\beta_{0,L}, \beta_{1,L}$ and $\beta_{0,R}, \beta_{1,R}$ such that the square-distance measure $D_2(A, u)$ is minimized, subject to the nonnegativity constraints on $\beta_{0,L}, \beta_{1,L}$ and $\beta_{0,R}, \beta_{1,R}$. Here u results to be a function of the $\underline{\beta}$= $(\beta_{0,L}, \beta_{1,L}, \beta_{0,R}, \beta_{1,R})$. Our problem is

$$Min \quad D_2(A, u(\underline{\beta})) s.t. \underline{\beta} \geq 0.$$

The distance $D_2(A, u(\underline{\beta}))$ can be calculated if we have other information on A. For example, if the membership function of A is known at other points, we can approximate the distance by a least squares functional.

A special case applies if we have available a single additional observation for the left side, say $\mu_A(x^-) = \mu^-$ with $x^- \in (a, b)$, $\mu^- \in (0, 1)$ and for the right, say $\mu_A(x^+) = \mu^+$ with $x^+ \in (c, d)$, $\mu^+ \in (0, 1)$. For example, the solution having least norm of (β_0, β_1) (i.e. with minimal $\beta_0^2 + \beta_1^2$), is obtained easily (see [4]).

Fig. 2 shows the solution with the following data: $a = 0$, $b = 2.5$, $c = 3$, $d = 5$ and the two membership functions for $\mu_A(0.5) = 0.5$, $\mu_A(3.5) = 0.9$ and $\mu_A(0.6) = 0.5$, $\mu_A(3.5) = 0.8$.

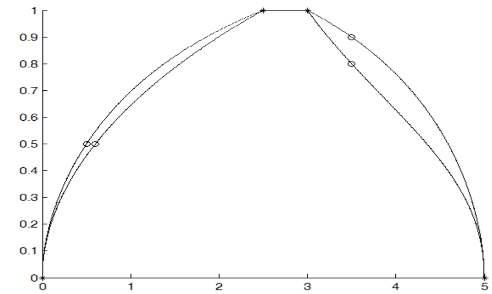

Figure 2: Least norm approximated solutions for two examples data.

4 Conclusion

We have suggested a parametrization of LR fuzzy numbers u by the use of parametric monotonic (simple) functions to model the left and the right branches of u. The obtained parametrizations form a subspace of the space of fuzzy numbers and can be possibly be refined to become a dense subspace. Within the parametrizations, we define approximated arithmetic operators for the basic arithmetic and for the fuzzy extension of functions by the application of Zadeh's extension principle.

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