2D Oculomotor Plant Mathematical Model for Eye Movement Simulation

Oleg V. Komogortsev, *Member, IEEE*, and Ukwatta K.S. Jayarathna, *Student Member, IEEE*

Abstract **-This paper builds a two dimensional Oculomotor Plant Mathematical Model (2DOPMM) that is capable of generating eye movement trace on a two dimensional plane. The key difference between the proposed model and the models presented previously is a design that is geared towards linearity and capability of integration into a real-time Human Computer Interaction system while providing force output for each extraocular muscle with values close to physiological measurements. The model is represented as a twelve order system created by a set of linear mechanical components representing major anatomical properties of extraocular muscles and the eye globe: muscle location, elasticity, viscosity, eye-globe rotational inertia, muscle active state tension, length tension and force velocity relationships. Linearity is a key point ensuring a real-time performance in an online implementation of the model with twelve order representation providing close match to the eye anatomical structure. Practical applications of the proposed model lie in the area of extraocular muscle effort estimation and Human Computer Interaction.**

***Keywords*- Oculomotor Plant, Eye Movements, Human Computer Interaction, Effort Estimation.**

I. INTRODUCTION

Oculomotor plant mathematical model (OPMM) development has provided an important impact on several scientific fields. For example, oculomotor plant models have been already employed in the medical field for the correction of the eye pathology called strabismus [1]. A horizontal OPMM was employed as a tool allowing reduction of sensor and transmission delays in the eye-gaze driven systems, through accurate eye movement prediction [2],[3]. Same OPMM was suggested as an extraocular muscle estimator for quantitative assessment of usability in graphical user interfaces [4]. It should be noted that muscular effort estimation appears to be an important problem in the medical community [5], [6]. Muscle effort is usually quantified by using the EMG signal analysis [5]. Such technique estimates effort as a normalized recorded activity of the EMG signal. An eye tracker can be employed to provide extraocular muscle force output through the use of the eye model presented in this paper. Eye tracking is particularly an attractive technology due to its non invasiveness. Modern eye trackers are represented by web camera-like devices without any parts affixed to the user's body.

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Oleg V. komogortsev is with the Department of Computer Science, Texas State University – San Marcos, TX 78666, USA. (e-mail: ok11@txstate.edu).

Ukwatta K.S. Jayarathna is with the Department of Computer Science, Texas State University – San Marcos, TX 78666, USA. (phone: 1-512-665-626293; e-mail: sampath@txstate.edu).

The contribution of the 2DOPMM presented in this paper is the ability to simulate two dimensional eye movements and to provide muscle force values for each extraocular muscle while maintaining linearity. Our model grows from a horizontal OPMM presented in [2], [3] by addition of two extraocular muscles (superior/inferior recti) and allowing vertical eye movements. When compared to three dimensional models [7], [8] our model accounts for individual anatomical components (passive elasticity, viscosity, eye-globe rotational inertia, muscle active state tension, length tension characteristics, force velocity relationships) and considers each extraocular muscle separately, therefore allowing estimation of individual extraocular muscle force values.

II. HUMAN VISUAL SYSTEM

This section provides the description for the anatomical apparatus involved in execution of basic eye movement types (fixations, saccades) and brainstem control mechanism responsible for extraocular muscle innervations.

A. Oculomotor Plant Mechanics

The eye globe rotates in its socket through the use of six muscles. These six muscles are the medial and the lateral recti – the muscles mainly responsible for horizontal eye movements; superior and inferior recti – the muscles mainly responsible for vertical eye movements; superior and inferior oblique – the muscles mainly responsible for eye rotations around its primary axis of sight; and vertical eye movements.

The brain sends a neuronal control signal to each muscle to direct the muscle to perform its work. A neuronal control signal is anatomically implemented as a neuronal discharge that is sent through a nerve to a designated muscle from the brain [9]. The frequency of this discharge determines the level of muscle innervations and results in a specific amount of work that a muscle will perform.

During an eye movement, movement trajectory can be separated into horizontal and vertical components. The neuronal control signal for the horizontal and the vertical components is generated by different parts of the brain. The control signal for the horizontal component is generated by the premotor neurons in the pons and medulla [9], and executed by the medial and the lateral recti muscles. The rostral midbrain generates a neuronal control signal for the vertical eye movement component [9]. The vertical eye movement component is executed by superior/inferior recti and superior/inferior oblique. During saccades the neuronal control signal for each muscle resembles a pulse-step function. The eye position during the onset of a saccade, the saccade's amplitude and direction define pulse and step parameters of the control signal. Once the parameters of the neuronal control signal are calculated by the brain, the control signal is sent as a neuronal discharge at the calculated frequency. During eye fixations neuronal discharge is performed at a constant rate that is linearly related to the eye position.

B. 2D Oculomotor Plant Mathematical Model

Our Oculomotor plant model stirred by the work of Komogortsev and Khan [2] where a linear horizontal oculomotor plant mechanical model was developed. The model consisted of the eye globe and two extraocular muscles: lateral and medial recti, and modeled only the horizontal movement of the right eye. Based on the anatomical properties of the eye muscle properties, we found the model can be easily extended to develop the eye movements with both horizontal and vertical components in them.

Our 2DOPMM is driven by the neuronal control signal innervating four extraocular muscles lateral, medial, superior and inferior recti that induce eye globe rotation. The 2DOPMM models resistive forces provided by surrounding tissues. The lateral, medial, superior and the inferior recti are modeled through a system of mechanical components described in latter subsections. Each muscle plays the role of the agonist or the antagonist. The agonist muscle contracts and pulls the eye globe in the required direction and the antagonist muscle stretches and resists the pull [2].

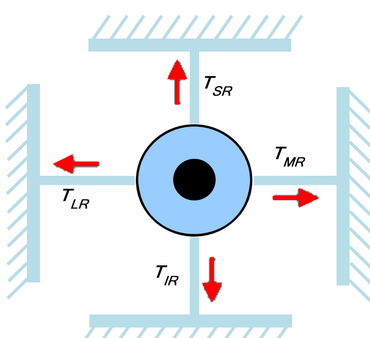

Fig. 1. Oculomotor Plant Mathematical Model with four muscle forces

Latter subsections describe the 2DOPMM based on the role that each muscle plays in a particular situation. We present a detailed example when the eye globe rotates Right Upward, and lateral/superior recti induce the movement and medial/inferior recti resist movement. We can say that lateral/superior recti play the agonist and medial/inferior recti play the antagonist role in this particular situation. Evoked by muscle movement, an eye can move in eight different directions: Right Horizontal, Left Horizontal, Top Vertical, Bottom Vertical, Right Upward, Left Upward, Right Downward and Left Downward. This paper discusses in detail only Right Upward and Left Downward movement as examples, but the model can be modified to simulate eye globe movements in all directions.

Our 2DOPMM is implemented with four contour points located around the eye globe. Each point is attached to the corresponding extraocular muscle inducing a required rotation of the eye through generation of forces T_{LR}, T_{MR}, T_{SR}, and T_{IR}. Fig. 1 presents the diagram of the 2DOPMM in its primary position with coordinates (0, 0). The subscript notation identifies LR with lateral rectus, MR with medial rectus, SR with superior rectus and IR with inferior rectus. Later in the paper the agonist muscle's parameters are identified with AG subscript (example N_{AG}, B_{AG}), antagonist with ANT. Subscripts HR and VR will represent horizontal and vertical components of the oculomotor plant respectively. The force direction of a particular muscle is represented by T (Top), B (Bottom), R (Right), and L (Left) subscript notations. In our work, Right and Top force directions are regarded as positive for eye movements in any direction. Subscript notation MF is used to represent the force applied by a muscle to the eye globe. Parameters without those subscripts are identical to both types of muscles.

C. Individual Muscle Properties

Detailed computation of muscle forces requires accurate modeling of each component inside of an extraocular muscle. These components are **passive elasticity**, an **active state tension**, **series elasticity**, a **length-tension component** and a **force velocity relationship** [10], [11], [12]. Combined these components create a Muscle Mechanical Model (MMM).

Passive Elasticity: Each body muscle in the rest state is elastic. The rested muscle can be stretched by applying force, the extension being proportional to the force applied. The passive muscle component is nonlinear, but in this paper it is modeled as an ideal linear spring. The numerical value for the spring coefficient representing passive elasticity was estimated by Collins [10] to be K_p=0.5 grams of tension per degree.

Active State Tension: Each muscle produces active state tension when it is stimulated. If stimulated by a single wave of neurons a muscle twitches then relaxes. A muscle goes into the tetanic state, when it is stimulated by neurons at a specific frequency continuously [13]. When a tetanic stimulation occurs, a muscle develops tension, trying to contract. The resulting tension is called the active state tension. The intensity of the active state tension depends upon the frequency of the neuronal discharge. An ideal force generator component is used in the MMM to represent active state tension F_{LR} with lateral rectus, F_{MR} with medial rectus, F_{IR} with inferior rectus, and F_{SR} with superior rectus.

Length Tension Relationship: The tension that a muscle develops as a result of neuronal stimulation partially depends on its length. In this paper the length tension relationship is modeled as an ideal linear spring. The linear coefficient K_{LT} of the spring was measured by Bahill [11] to be 1.2 grams of tension per degree.

Series Elasticity: The series elasticity is in series with the active force generator, hence the name. In the MMM, the series elasticity component is modeled as an ideal linear spring. The linear coefficient K_{SE} of the spring was measured by Collins [10] to be 2.5 grams of tension per degree.

Force Velocity Relationship: This relationship shows that a muscle is capable of producing larger forces at lower velocities. This dependency of force upon velocity varies for different levels of a neuronal control signal and depends on whether a muscle shortens or being stretched. In this research, the velocity of muscle contraction is connected to a change of length in the length tension component of the muscle and represented by the following variables employed either in horizontal or vertical plane: $\Delta\theta_{plane\_LT\_LR}$ - lateral rectus, $\Delta\theta_{plane\_LT\_MR}$- medial rectus, $\Delta\theta_{plane\_LT\_SR}$- superior rectus, $\Delta\theta_{plane\_LT\_IR}$- inferior rectus. The values for the viscous elements are, B_{AG} = 0.04 grams-sec/° and B_{ANT}= 0.02 grams-sec [11].

III. RIGHT UPWARD EYE MOVEMENT

When human eye rotates to a new target axis due to visual stimuli, each eye muscle is innervated by a neuronal control signal producing necessary muscle forces for eye globe rotation with required direction and magnitude. In this subsection we will compute Horizontal Right, Horizontal Left, Vertical Top and Vertical Bottom muscle forces. Similar to each extraocular muscle, these forces can be classified as agonist or antagonist. Computation of these forces will allow us computing individual extraocular muscle forces later.

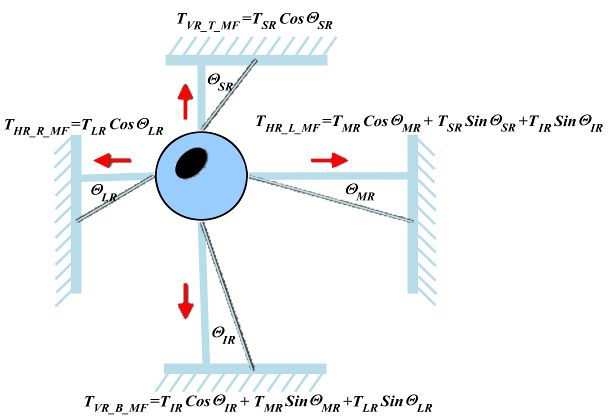

Fig. 2. Right Upward eye movement with vertically and horizontally projected muscle forces.

During Right Upward movement of the right eye (eye executing a saccade between two fixations) the lateral rectus and the superior rectus as agonists move the eye to its destination stretching the medial and inferior recti. New destination (fixation point) results in equilibrium of extraocular muscle forces and the forces associated with the eye globe. Θ_{HR} represents horizontal component of eye globe rotation measured in degrees while Θ_{VR} represents vertical component. Each of the four muscles of our 2DOPMM becomes tilted to a respective angles, Θ_{LR}, Θ_{SR}, Θ_{MR}, *and* Θ_{IR}.

When mapped to the horizontal and vertical planes, each extraocular muscle provides a projection of its force. Those projections are directed according to original two dimensional movements, therefore creating four equations of forces in horizontal and vertical planes:

Horizontal Right Muscle Force ($T_{HR\_R\_MF}$): $T_{LR}Cos\Theta_{LR}$
Horizontal Left Muscle Force ($T_{HR\_L\_MF}$) : $T_{MR}Cos\Theta_{MR}$ + $T_{SR}Sin\Theta_{SR}$+ $T_{IR}Sin\Theta_{IR}$
Vertical Top Muscle Force ($T_{VR\_T\_MF}$) : $T_{SR}Cos\Theta_{SR}$
Vertical Bottom Muscle Force ($T_{VR\_B\_MF}$) : $T_{IR}Cos\Theta_{IR}$ + $T_{MR}Sin\Theta_{MR}$ + $T_{LR}Sin\Theta_{LR}$

A. Horizontal Right Muscle Force ($T_{HR\_R\_MF}$)

During Right Upward eye movement Right Muscle Force *(HR_R_MF)*, pulls the eye-globe in positive direction and can be classified as the agonist. According to diagram presented in Fig. 2, lateral rectus is the only muscle contributing to the $T_{HR\_R\_MF}$ force.

Lets assume that prior to the eye movement, the length of the displacement in the series elasticity and the length tension spring components in the horizontal direction added together was θ_{HR_LR}. Considering that the right eye moves to the right by $\Delta\theta_{HR}$ the original θ_{HR_LR} displacement in the lateral rectus is reduced making the resulting displacement θ_{HR_LR} - $\Delta\theta_{HR\_LR}$. The displacement $\Delta\theta_{HR\_LR}$ can be broken into the displacements inside of series elastic and the length tension componenets as: $\Delta\theta_{HR\_LR}$=$\Delta\theta_{HR\_SE\_LR}$ - $\Delta\theta_{HR\_LT\_LR}$. Muscle contraction expands the series elastic component making the resulting displacement $\theta_{HR\_SE\_LR}$+$\Delta\theta_{HR\_SE\_LR}$. Muscle contraction shortens the length tension component making the resulting displacement $\theta_{HR\_LT\_LR}$-$\Delta\theta_{HR\_LT\_LR}$. The damping component modeling the force velocity relationship $B_{AG}\Delta\dot{\theta}_{HR\_LT\_LR}$ resists the muscle contraction. The amount of resistive force produced by the damping component is based upon the velocity of contraction of the length tension component.

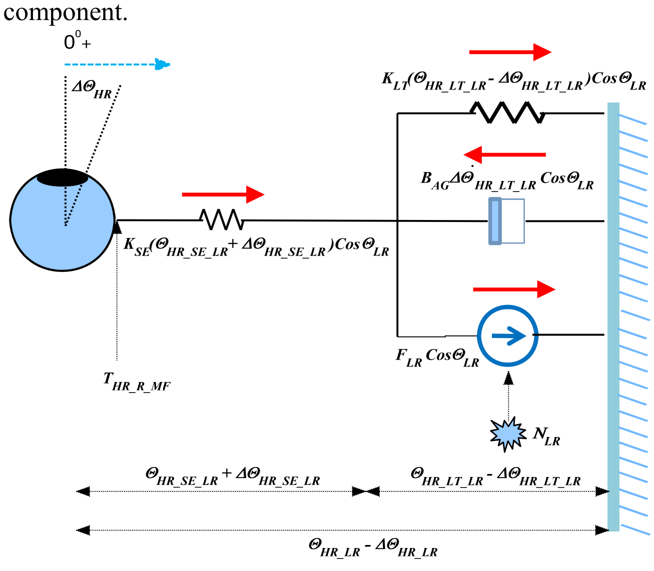

Fig. 3. Horizontal Right Muscle force. Forces generated by individual MMM components are marked with arrows.

Using Fig. 3, we can write the equation of force with which the part of the diagram responsible for contraction by the lateral rectus (active state tension, damping component, length tension component) pulls the series elasticity component.

$$T_{HR\_R\_MF} = F_{LR}\cos\theta_{LR} + K_{LT}(\theta_{HR\_LT\_LR} - \Delta\theta_{HR\_LT\_LR})\cos\theta_{LR} - B_{AG}\Delta\dot{\theta}_{HR\_LT\_LR}\cos\theta_{LR} \quad \textbf{(1)}$$

Resisting the contraction, the series elasticity components of lateral rectus propagates the contractile force by pulling the eye globe with the same force $T_{HR\_R\_MF}$.

$$T_{HR\_R\_MF} = K_{SE}(\theta_{HR\_SE\_LR} + \Delta\theta_{HR\_SE\_LR}) \cos\theta_{LR} \tag{2}$$

Equations (1), and (2), can be used to calculate the force $T_{HR\_R\_MF}$ in terms of the eye rotation $\Delta\theta_{HR}$ and displacement $\Delta\theta_{HR\_LT\_LR}$ of the length tension component of the muscle [14].

$$T_{HR\_R\_MF} = \frac{\hat{F}_{LR}K_{SE}}{K_{SE}+K_{LT}} - \frac{\Delta\theta_{HR}K_{SE}}{K_{SE}+K_{LT}} - \hat{B}_{AG}\Delta\dot{\theta}_{HR\_LT\_LR} \tag{3}$$

Where, $\hat{F}_{LR} = F_{LR} - \mathrm{K_{SE}}(\theta_{HR\_LR} - \theta_{HR\_LT\_LR}) + K_{LT}\theta_{HR\_LT\_LR}$

$$T_{HR\_R\_MF} = K_{SE}(\Delta\theta_{HR\_LT\_LR} - \Delta\theta_{HR}) \tag{4}$$

Fig. 3 presents a detailed diagram of the lateral rectus pulling the eye globe during Right Upward rotation.

B. *Horizontal Left Muscle Force ($T_{HR\_L\_MF}$)*

Three muscles, inferior (antagonist), medial (antagonist) and superior (agonist) recti contribute to the Horizontal Left Muscle force which can be classified as the antagonist.

Considering movement dynamics of medial rectus, we can say that prior to the eye rotation, the length of the displacement in the series elasticity and the length tension springs added together was θ_{HR_MR}. Horizontal eye rotation by $\Delta\theta_{HR}$ degrees causes change in the displacement θ_{HR_MR} by increasing it by $\Delta\theta_{HR\_MR}$ making the resulting displacement $\theta_{HR\_MR} + \Delta\theta_{HR\_MR}$. Both length tension and series elasticity components lengthen as a result of the agonist pull by the horizontal right force. $\Delta\theta_{HR\_MR}$ can be split into the displacement of the series elasticity component and the length tension component: $\Delta\theta_{HR\_MR}=\Delta\theta_{HR\_SE\_MR}+\Delta\theta_{HR\_LT\_MR}$. The resulting displacement for the series elasticity component is $\theta_{HR\_SE\_MR}+\Delta\theta_{HR\_SE\_MR}$ and for the length tension component is $\theta_{HR\_LT\_MR}+\Delta\theta_{HR\_LT\_MR}$. The damping component modeling the force velocity relationship $B_{ANT}\Delta\dot{\theta}_{HR\_LT\_MR}$ resists the muscle stretching. The amount of resistive force is based upon the velocity of stretching of the length tension component.

During Right Upward rotation, inferior rectus MMM mirros the behaviour of the medial rectus but behaves in the vertical plane.

During Right Upward movement, dynamics inside of superior rectus are as follows: as result of the verical eye rotation by $\Delta\theta_{VR}$ degrees, the original displacement θ_{VR_SR} is reduced to $\theta_{VR\_SR} - \Delta\theta_{SR}$. The displacement $\Delta\theta_{SR}$ can be broken into $\Delta\theta_{SR} = \Delta\theta_{VR\_SE\_SR} - \Delta\theta_{VR\_LT\_SR}$. Muscle contraction expands the series elastic component making the resulting displacement $\theta_{VR\_SE\_SR}+\Delta\theta_{VR\_SE\_SR}$. Muscle contraction shortens the length tension component making the resulting displacement $\theta_{VR\_LT\_SR} - \Delta\theta_{VR\_LT\_SR}$. The damping component modeling the force velocity relationship $B_{AG}\Delta\dot{\theta}_{VR\_LT\_SR}$ resists the muscle contraction. The amount of resistive force produced by the damping component is based upon the velocity of contraction of the length tension component [14]. We can write equation of forces with which the part of the diagram responsible for the contraction (active state tensions, damping components, length tension components) pulls the series elasticity components,

$$T_{MR} = -F_{MR}\cos\theta_{MR} - K_{LT}(\theta_{HR\_LT\_MR} + \Delta\theta_{HR\_LT\_MR})\cos\theta_{MR} - B_{ANT}\Delta\dot{\theta}_{HR\_LT\_MR}\cos\theta_{MR} \tag{5}$$

$$T_{IR} = -F_{IR}\sin\theta_{IR} - K_{LT}(\theta_{VR\_LT\_IR} + \Delta\theta_{VR\_LT\_IR})\sin\theta_{IR} - B_{ANT}\Delta\dot{\theta}_{VR\_LT\_IR}\sin\theta_{IR} \tag{6}$$

$$T_{SR} = -F_{SR}\sin\theta_{SR} - K_{LT}(\theta_{VR\_LT\_SR} - \Delta\theta_{VR\_LT\_SR})\sin\theta_{SR} + B_{AG}\Delta\dot{\theta}_{VR\_LT\_SR}\sin\theta_{SR} \tag{7}$$

$$T_{MR} = -K_{SE}(\theta_{HR\_SE\_MR} + \Delta\theta_{HR\_SE\_MR})\cos\theta_{MR} \tag{8}$$

$$T_{IR} = -K_{SE}(\theta_{VR\_SE\_IR} + \Delta\theta_{VR\_SE\_IR})\sin\theta_{IR} \tag{9}$$

$$T_{SR} = -K_{SE}(\theta_{VR\_SE\_SR} + \Delta\theta_{VR\_SE\_SR})\sin\theta_{SR} \tag{10}$$

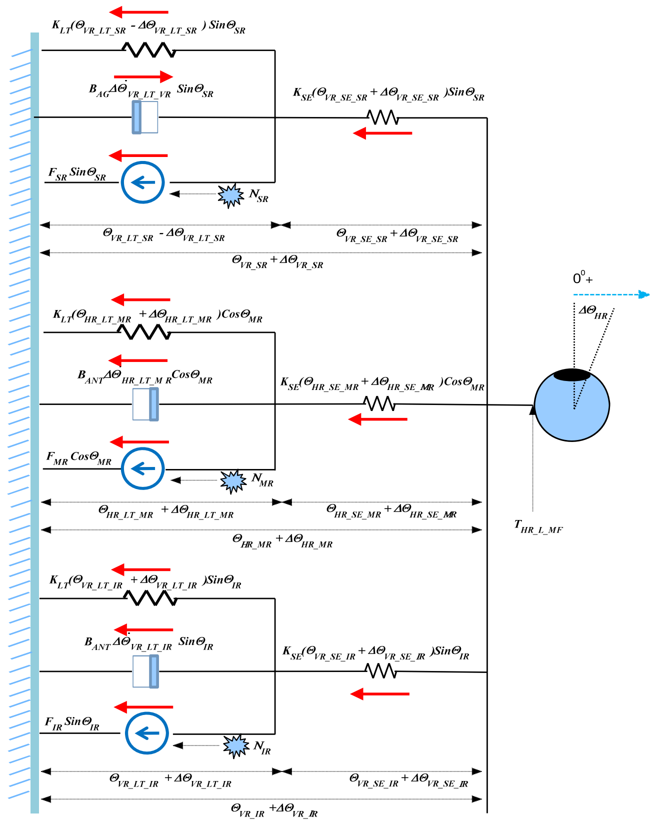

Fig. 4. Horizontal Left Muscle force. Forces generated by individual MMM components are marked with arrows.

Equations (5), (6), (7), (8), (9), and (10) can be used to calculate forces T_{MR}, T_{IR}, and T_{SR}, in terms of the eye rotation $\Delta\theta_{HR}$, $\Delta\theta_{VR}$ and displacements $\Delta\theta_{HR\_LT\_MR}$, $\Delta\theta_{VR\_LT\_IR}$, $\Delta\theta_{VR\_LT\_SR}$ of the length tension components of each muscle respectively [14].

$$T_{MR} = -\frac{\hat{F}_{MR}K_{SE}}{K_{SE}+K_{LT}} - \frac{\Delta\theta_{HR}K_{SE}}{K_{SE}+K_{LT}} - \hat{B}_{ANT}\Delta\dot{\theta}_{HR\_LT\_MR} \tag{11}$$

Where, $\hat{F}_{MR} = F_{MR} + \mathrm{K_{SE}}(\theta_{HR\_LT\_MR} - \theta_{HR\_MR}) + K_{LT}\theta_{HR\_LT\_MR}$

$$T_{MR} = -\mathrm{K_{SE}}(\Delta\theta_{HR} - \Delta\theta_{HR\_LT\_MR}) \tag{12}$$

$$T_{IR} = -\frac{\hat{F}_{IR}K_{SE}}{K_{SE}+K_{LT}} - \frac{\Delta\theta_{VR}K_{SE}}{K_{SE}+K_{LT}} - \hat{B}_{ANT}\Delta\dot{\theta}_{VR\_LT\_IR} \tag{13}$$

Where, $\hat{F}_{IR} = F_{IR} + \mathrm{K_{SE}}(\theta_{VR\_LT\_IR} - \theta_{VR\_IR}) + K_{LT}\theta_{VR\_LT\_IR}$

$$T_{IR} = -\mathrm{K_{SE}}(\Delta\theta_{VR} - \Delta\theta_{VR\_LT\_IR}) \tag{14}$$

$$T_{SR} = -\frac{\hat{F}_{SR}K_{SE}}{K_{SE}+K_{LT}} + \frac{\Delta\theta_{VR}K_{SE}}{K_{SE}+K_{LT}} + \hat{B}_{AG}\Delta\dot{\theta}_{VR\_LT\_SR} \tag{15}$$

Where, $\hat{F}_{SR} = F_{SR} - \mathrm{K_{SE}}(\theta_{VR\_SR} - \theta_{VR\_LT\_SR}) + K_{LT}\theta_{VR\_LT\_SR}$

$$T_{SR} = -K_{SE}(\Delta\theta_{VR\_LT\_SR} - \Delta\theta_{VR}) \quad \textbf{(16)}$$

By using equations (11) (12) ,(13) , (14), (15) (16) on each muscle forces T_{MR} , T_{IR}, T_{SR} we can formulate the $T_{HR\_L\_MF}$ in the following way [14].

$$T_{HR\_L\_MF} = T_{MR} + T_{IR} + T_{SR}$$

Using equations (11), (13), and (15),

$$T_{HR\_L\_MF} = -\frac{(\hat{F}_{MR} + \hat{F}_{IR} + \hat{F}_{SR})}{K_{SE} + K_{LT}} K_{SE} - \frac{\Delta\theta_{HR} K_{SE}}{K_{SE} + K_{LT}} - \hat{B}_{ANT}\Delta\dot{\theta}_{HR\_LT\_MR} - \hat{B}_{ANT}\Delta\dot{\theta}_{VR\_LT\_IR} + \hat{B}_{AG}\Delta\dot{\theta}_{VR\_LT\_SR} \quad \textbf{(17)}$$

Using equations (12), (14), and (16),

$$T_{HR\_L\_MF} = -K_{SE}(\Delta\theta_{HR} - \Delta\theta_{HR\_LT\_MR} - \Delta\theta_{VR\_LT\_IR} + \Delta\theta_{VR\_LT\_SR}) \quad \textbf{(18)}$$

C. *Vertical Top Muscle Force ($T_{VR\_T\_MF}$)*

In Right Upward movement, the Vertical Top Muscle Force can be classified as the agonist. According to Fig. 2, the only muscle contributing to the $T_{VR\_T\_MF}$ is superior rectus. Superior rectus MMM works in the vertical plane with similar dynamics as MMM of lateral rectus. Following the logic presented for lateral rectus in Section II.A (Horizontal Right Muscle Force section) equations describing the MMM of the superior rectus can be derived. For detailed calculations refer to [14].

$$T_{VR\_T\_MF} = \frac{\hat{F}_{SR} K_{SE}}{K_{SE} + K_{LT}} - \frac{\Delta\theta_{VR} K_{SE}}{K_{SE} + K_{LT}} - \hat{B}_{AG}\Delta\dot{\theta}_{VR\_LT\_SR} \quad \textbf{(19)}$$

Where, $\hat{F}_{SR} = F_{SR} - K_{SE}(\theta_{VR\_SR} - \theta_{VR\_LT\_SR}) + K_{LT}\theta_{VR\_LT\_SR}$

$$T_{VR\_T\_MF} = K_{SE}(\Delta\theta_{VR\_LT\_SR} - \Delta\theta_{VR}) \quad \textbf{(20)}$$

D. *Vertical Bottom Muscle Force ($T_{VR\_B\_MF}$)*

Assuming that the projecttions of medial rectus, and inferor rectus play the role of the antagonist, and lateral rectus plays the agonist, the MMM of the Vertical Bottom Muscle Force in the positive direction resembles with Fig. 4, muscles replaced by appropreate subscript notations and movement in an appropreate horizontal or/and vertical plane.

This shows a similar formation of muscle forces T_{MR}, and T_{IR} as in horizontal left muscle force section and T_{LR} as in horizontal right muscle force section with opposite sign value, and in all three muscle forces, changing the muscle angles from cosine to sine or sine to cosine. T_{LR} in equations (25), and (26), shows opposite sign values of the equations (1) and (2), because they are in the bottom force direction which is considered as negative in this research.

$$T_{MR} = -F_{MR}\sin\theta_{MR} - K_{LT}(\theta_{HR\_LT\_MR} + \Delta\theta_{HR\_LT\_MR})\sin\theta_{MR} - B_{ANT}\Delta\dot{\theta}_{HR\_LT\_MR}\sin\theta_{MR} \quad \textbf{(21)}$$

$$T_{MR} = -K_{SE}(\theta_{HR\_SE\_MR} + \Delta\theta_{HR\_SE\_MR})\sin\theta_{MR} \quad \textbf{(22)}$$

$$T_{IR} = -F_{IR}\cos\theta_{IR} - K_{LT}(\theta_{VR\_LT\_IR} + \Delta\theta_{VR\_LT\_IR})\cos\theta_{IR} - B_{ANT}\Delta\dot{\theta}_{VR\_LT\_IR}\cos\theta_{IR} \quad \textbf{(23)}$$

$$T_{IR} = -K_{SE}(\theta_{VR\_SE\_IR} + \Delta\theta_{VR\_SE\_IR})\cos\theta_{IR} \quad \textbf{(24)}$$

$$T_{LR} = -F_{LR}\sin\theta_{LR} - K_{LT}(\theta_{HR\_LT\_LR} - \Delta\theta_{HR\_LT\_LR})\sin\theta_{LR} + B_{AG}\Delta\dot{\theta}_{HR\_LT\_LR}\sin\theta_{LR} \quad \textbf{(25)}$$

$$T_{LR} = -K_{SE}(\theta_{HR\_SE\_LR} + \Delta\theta_{HR\_SE\_LR})\sin\theta_{LR} \quad \textbf{(26)}$$

By equations (21), and (22), we can calculate the muscle force T_{MR} which is as same as equations (11), and (12).

$$T_{MR} = -\frac{\hat{F}_{MR} K_{SE}}{K_{SE} + K_{LT}} - \frac{\Delta\theta_{HR} K_{SE}}{K_{SE} + K_{LT}} - \hat{B}_{ANT}\Delta\dot{\theta}_{HR\_LT\_MR} \quad \textbf{(27)}$$

Where, $\hat{F}_{MR} = F_{MR} + K_{SE}(\theta_{HR\_LT\_MR} - \theta_{HR\_MR}) + K_{LT}\theta_{HR\_LT\_MR}$

$$T_{MR} = -K_{SE}(\Delta\theta_{HR} - \Delta\theta_{HR\_LT\_MR}) \quad \textbf{(28)}$$

By equations (23), and (24), we can calculate the muscle force T_{IR} which is as same as equations (13), and (14).

$$T_{IR} = -\frac{\hat{F}_{IR} K_{SE}}{K_{SE} + K_{LT}} - \frac{\Delta\theta_{VR} K_{SE}}{K_{SE} + K_{LT}} - \hat{B}_{ANT}\Delta\dot{\theta}_{VR\_LT\_IR} \quad \textbf{(29)}$$

Where, $\hat{F}_{IR} = F_{IR} + K_{SE}(\theta_{VR\_LT\_IR} - \theta_{VR\_IR}) + K_{LT}\theta_{VR\_LT\_IR}$

$$T_{IR} = -K_{SE}(\Delta\theta_{VR} - \Delta\theta_{VR\_LT\_IR}) \quad \textbf{(30)}$$

By equations (25), and (26), we can calculate T_{LR} which is the same except with opposite sign as of equations (3), and (4).

$$T_{LR} = -\frac{\hat{F}_{LR} K_{SE}}{K_{SE} + K_{LT}} + \frac{\Delta\theta_{HR} K_{SE}}{K_{SE} + K_{LT}} + \hat{B}_{AG}\Delta\dot{\theta}_{HR\_LT\_LR} \quad \textbf{(31)}$$

Where, $\hat{F}_{LR} = F_{LR} - K_{SE}(\theta_{HR\_LR} - \theta_{HR\_LT\_LR}) + K_{LT}\theta_{HR\_LT\_LR}$

$$T_{LR} = -K_{SE}(\Delta\theta_{HR\_LT\_LR} - \Delta\theta_{HR}) \quad \textbf{(32)}$$

By using equations (27), (28), (29), (30), (31), and (32) on each muscle forces T_{MR} , T_{IR}, and T_{LR} we can formulate the $T_{VR\_B\_MF}$,

$$T_{VR\_B\_MF} = T_{MR} + T_{IR} + T_{LR}$$

Using equations (27), (29), and (31),

$$T_{VR\_B\_MF} = -\frac{(\hat{F}_{MR} + \hat{F}_{IR} + \hat{F}_{LR})}{K_{SE} + K_{LT}} K_{SE} - \frac{\Delta\theta_{VR} K_{SE}}{K_{SE} + K_{LT}} - \hat{B}_{ANT}\Delta\dot{\theta}_{HR\_LT\_MR} - \hat{B}_{ANT}\Delta\dot{\theta}_{VR\_LT\_IR} + \hat{B}_{AG}\Delta\dot{\theta}_{HR\_LT\_LR} \quad \textbf{(33)}$$

Using equations (28), (30), and (32),

$$T_{VR\_B\_MF} = -K_{SE}(\Delta\theta_{VR} - \Delta\theta_{VR\_LT\_IR} - \Delta\theta_{HR\_LT\_MR} + \Delta\theta_{HR\_LT\_LR}) \quad \textbf{(34)}$$

E. *Active State Tension*

Active state tension appears as a result of the neuronal control signal sent by the brain. Though the neuronal control signals $N_{AG}(t)$, $N_{ANT}(t)$, for the agonist and the antagonist muscles rise and drop instantaneously, neither the forces that muscles apply to the eye globe nor active state tensions rise to their maximum values immediately [11]. This happens due to the anatomical characteristics of the neuronal signaling.

In our 2DOPMM model, the active state tension is a result of a low pass filtering process performed upon the neuronal control signal. Active state tension dynamics can be represented with the following differential equations at each time interval both in horizontal and vertical planes [14],

$$\dot{\hat{F}}_{LR}(t) = \frac{N_{LR} - \hat{F}_{LR}(t)}{\tau_{LR}} \quad \textbf{(35)}$$

$$\dot{\hat{F}}_{MR}(t) = \frac{N_{MR} - \hat{F}_{MR}(t)}{\tau_{MR}} \quad \textbf{(36)}$$

$$\dot{\hat{F}}_{SR}(t) = \frac{N_{SR} - \hat{F}_{SR}(t)}{\tau_{SR}} \quad \textbf{(37)}$$

$$\dot{\hat{F}}_{IR}(t) = \frac{N_{IR} - \hat{F}_{IR}(t)}{\tau_{IR}} \quad \textbf{(38)}$$

τ_{LR}, τ_{MR}, τ_{SR}, τ_{IR}, are functions that define the low pass filtering process; they are defined by the activation and deactivation time constants that are selected empirically to match human physiological data [11].

IV. LEFT DOWNWARD EYE MOVEMENT

In our Right Upward eye movement model, we found that the eye movement in the Right Upward direction can be explicitly expressed using eight model equations. In this section we extended the knowledge of Right Upward model to build equations for Left Downward eye movement, which can be used to compare equations and generalize four model equations to identify eye movement in any direction.

Consider a Left Downward movement of the right eye, which moves the eye visual axis from the initial coordination position (0, 0). The medial rectus and the inferior rectus contracts and provide forces to rotate the eye globe to its destination. The muscle lateral rectus and superior rectus stretches and contest the movement in the required direction. When the eye fixates upon the visual target, agonist muscles forces are compensated by the antagonist muscle forces and the elastic/viscous properties of the eye globe, as in Right Upward eye movement. During Left Downward eye movement, medial, inferior rectus muscles play the agonist role and lateral, superior muscles play the antagonist role.

Each muscle provides projections of its forces activated by the neuronal control signal, at each point in the eye globe in the Left Downward eye movement. The projections of each muscle force is directed according to the force direction of its basic movement and this provides four basic muscle force equations at each point of connection at the eye globe as,

Horizontal Left Muscle Force ($T_{HR\_L\_MF}$) : $T_{MR}Cos\Theta_{MR}$
Horizontal Right Muscle Force ($T_{HR\_R\_MF}$) : $T_{LR}Cos\Theta_{LR}$ + $T_{SR}Sin\Theta_{SR}$+ $T_{IR}Sin\Theta_{IR}$
Vertical Bottom Muscle Force ($T_{VR\_B\_MF}$) : $T_{IR}Cos\Theta_{IR}$
Vertical Top Muscle Force ($T_{VR\_T\_MF}$) : $T_{SR}Cos\Theta_{SR}$ + $T_{MR}Sin\Theta_{MR}$ + $T_{LR}Sin\Theta_{LR}$

Each of four muscle forces; Horizontal Left, Horizontal Right, Vertical Top and Vertical Bottom consisist of two equations, each describing its formation as in the section describing the Right Upward eye movement. Right Upward eye movement equations (3), (4), (17), (18), (19), (20), (33), (34), each illustrates the eye movements Horizontal Right, Horizontal Left, Vertical Top and Vertical Bottom muscle forces respectively.

By applying the same methodology as in Right Upward eye movement, we can obtain eight more equations in order to express the eye movement in the Left Downward [14].

$$T_{HR\_L\_MF} = -K_{SE}(\Delta\theta_{HR\_LT\_MR} - \Delta\theta_{HR}) \quad \textbf{(39)}$$

$$T_{HR\_L\_MF} = -\frac{\hat{F}_{MR}K_{SE}}{K_{SE}+K_{LT}} + \frac{\Delta\theta_{HR}K_{SE}}{K_{SE}+K_{LT}} + \hat{B}_{AG}\Delta\dot{\theta}_{HR\_LT\_MR} \quad \textbf{(40)}$$

$$T_{HR\_R\_MF} = K_{SE}(\Delta\theta_{HR} - \Delta\theta_{HR\_LT\_LR} + \Delta\theta_{VR\_LT\_IR} - \Delta\theta_{VR\_LT\_SR}) \quad \textbf{(41)}$$

$$T_{HR\_R\_MF} = \frac{(\hat{F}_{LR}+\hat{F}_{IR}+\hat{F}_{SR})}{K_{SE}+K_{LT}}K_{SE} + \hat{B}_{ANT}\Delta\dot{\theta}_{HR\_LT\_LR} - \hat{B}_{AG}\Delta\dot{\theta}_{VR\_LT\_IR} + \hat{B}_{ANT}\Delta\dot{\theta}_{VR\_LT\_SR} \quad \textbf{(42)}$$

$$T_{VR\_B\_MF} = -K_{SE}(\Delta\theta_{VR\_LT\_IR} - \Delta\theta_{VR}) \quad \textbf{(43)}$$

$$T_{VR\_B\_MF} = -\frac{\hat{F}_{IR}K_{SE}}{K_{SE}+K_{LT}} + \frac{\Delta\theta_{VR}K_{SE}}{K_{SE}+K_{LT}} + \hat{B}_{AG}\Delta\dot{\theta}_{VR\_LT\_IR} \quad \textbf{(44)}$$

$$T_{VR\_T\_MF} = K_{SE}(\Delta\theta_{VR} - \Delta\theta_{VR\_LT\_SR} + \Delta\theta_{HR\_LT\_MR} - \Delta\theta_{HR\_LT\_LR}) \quad \textbf{(45)}$$

$$T_{VR\_T\_MF} = \frac{(\hat{F}_{LR}+\hat{F}_{MR}+\hat{F}_{SR})}{K_{SE}+K_{LT}}K_{SE} + \frac{\Delta\theta_{VR}K_{SE}}{K_{SE}+K_{LT}} + \hat{B}_{ANT}\Delta\dot{\theta}_{HR\_LT\_LR} - \hat{B}_{AG}\Delta\dot{\theta}_{HR\_LT\_MR} + \hat{B}_{ANT}\Delta\dot{\theta}_{VR\_LT\_SR} \quad \textbf{(46)}$$

V. RESULTS

A. OPMM equations (Right Upward eye movement)

The dynamics of the Right Upward eye movement can be described through a set of equations responsible for the vertical component of the movement and the horizontal component of the movement.

We describe horizontal component of the movement by six differential equations. First equation relates to the dynamics of the Right Muscle Force and can be derived by combining (3), and (4),

$$K_{SE}(\Delta\theta_{HR\_LT\_LR} - \Delta\theta_{HR}) = \frac{\hat{F}_{LR}K_{SE}}{K_{SE}+K_{LT}} - \frac{\Delta\theta_{HR}K_{SE}}{K_{SE}+K_{LT}} - \hat{B}_{AG}\Delta\dot{\theta}_{HR\_LT\_LR} \quad \textbf{(47)}$$

Second equation relates to the Left Muscle Force. Equations (17), and (18), can be combined together,

$$K_{SE}(\Delta\theta_{HR} - \Delta\theta_{HR\_LT\_MR} - \Delta\theta_{VR\_LT\_IR} + \Delta\theta_{VR\_LT\_SR}) = \frac{(\hat{F}_{MR}+\hat{F}_{IR}+\hat{F}_{SR})}{K_{SE}+K_{LT}}K_{SE} + \frac{\Delta\theta_{HR}K_{SE}}{K_{SE}+K_{LT}} + \hat{B}_{ANT}\Delta\dot{\theta}_{HR\_LT\_MR} + \hat{B}_{ANT}\Delta\dot{\theta}_{VR\_LT\_IR} - \hat{B}_{AG}\Delta\dot{\theta}_{VR\_LT\_SR} \quad \textbf{(48)}$$

According to Newton's second law, the sum of all forces acting on the eye globe equals the acceleration of the eye globe multiplied by the inertia of the eye globe. We can apply this law to horizontal and vertical component of movement separately.

$$J\Delta\ddot{\theta}_{HR} = T_{HR\_R\_MF} - T_{HR\_L\_MF} - K_p\Delta\theta_{HR} - B_p\Delta\dot{\theta}_{HR}$$

J - Eye globe's inertia, $\Delta\theta_{HR}$ – horizontal velocity of eye rotation, $\Delta\dot{\theta}_{HR}$ velocity of the eye rotation, $\Delta\ddot{\theta}_{HR}$ eye rotation acceleration. B_p=0.06 grams-s/degrees – viscosity of the tissues around the eye globe with Fig. 5 presents the case when viscous and passive elastic forces are added to the 2DOPMM for horizontal and vertical component of movement. $T_{HR\_R\_MF,}$, and $T_{HR\_L\_MF}$ can be calculated by equations (4) and (18) respectively:

$$J\Delta\ddot{\theta}_{HR} = K_{SE}(\Delta\theta_{HR\_LT\_LR} - \Delta\theta_{HR}) - K_{SE}(\Delta\theta_{HR} - \Delta\theta_{HR\_LT\_MR} - \Delta\theta_{VR\_LT\_IR} + \Delta\theta_{VR\_LT\_SR}) - K_p\Delta\theta_{HR} - B_p\Delta\dot{\theta}_{HR} \quad \textbf{(49)}$$

Following the logic presented above, we can write equations related to the Vertical Top Muscle Force by combining (19), and (20) and Vertical Bottom Muscle Force by (33), and (34).

$$K_{SE}(\Delta\theta_{VR\_LT\_SR} - \Delta\theta_{VR}) = \frac{\hat{F}_{SR}K_{SE}}{K_{SE}+K_{LT}} - \frac{\Delta\theta_{VR}K_{SE}}{K_{SE}+K_{LT}} - \hat{B}_{AG}\Delta\dot{\theta}_{VR\_LT\_SR} \quad \textbf{(50)}$$

$$\begin{aligned} K_{SE}(\Delta\theta_{VR} - \Delta\theta_{VR\_LT\_IR} - \Delta\theta_{HR\_LT\_MR} + \Delta\theta_{HR\_LT\_LR}) \\ = \frac{(\hat{F}_{MR}+\hat{F}_{IR}+\hat{F}_{LR})}{K_{SE}+K_{LT}}K_{SE} + \frac{\Delta\theta_{VR}K_{SE}}{K_{SE}+K_{LT}} \\ + \hat{B}_{ANT}\Delta\dot{\theta}_{HR\_LT\_MR} + \hat{B}_{ANT}\Delta\dot{\theta}_{VR\_LT\_IR} \\ - \hat{B}_{AG}\Delta\dot{\theta}_{HR\_LT\_LR} \end{aligned} \quad \textbf{(51)}$$

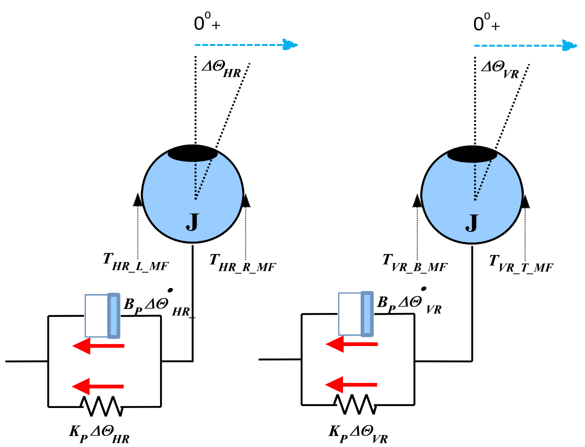

Fig. 5. Eye globe's inertia, viscosity of the tissues around eye globe and passive elasticity.

Applying Newton's second law we write the equation for the vertical component of movement involving eye globe inertia and forces acting on the eye globe.

$$J\Delta\ddot{\theta}_{VR} = T_{VR\_T\_MF} - T_{VR\_B\_MF} - K_p\Delta\theta_{VR} - B_p\Delta\dot{\theta}_{VR}$$

$T_{VR\_T\_MF}$, can be calculated from equation (20), and $T_{VR\_B\_MF}$ can be calculated from (34). Thus equation can be transformed into:

$$\begin{aligned} J\Delta\ddot{\theta}_{VR} = K_{SE}(\Delta\theta_{VR\_LT\_SR} - \Delta\theta_{VR}) \\ - K_{SE}(\Delta\theta_{VR} - \Delta\theta_{VR\_LT\_IR} - \Delta\theta_{HR\_LT\_MR} \\ + \Delta\theta_{HR\_LT\_LR}) - K_p\Delta\theta_{VR} - B_p\Delta\dot{\theta}_{VR} \end{aligned} \quad \textbf{(52)}$$

Two more differential equations can be added as:

$$\Delta\dot{\theta}_{HR} = \Delta\dot{\theta}_{HR} \quad \textbf{(53)}$$

$$\Delta\dot{\theta}_{VR} = \Delta\dot{\theta}_{VR} \quad \textbf{(54)}$$

Assigning $x_1 = \Delta\theta_{HR}$,$x_2 = \Delta\theta_{VR}$, $x_3 = \Delta\theta_{HR\_LT\_LR}$, $x_4 = \Delta\theta_{HR\_LT\_MR}$, $x_5 = \Delta\theta_{VR\_LT\_SR}$, $x_6 = \Delta\theta_{VR\_LT\_IR}$, $x_7 = \Delta\dot{\theta}_{HR}$, $x_8 = \Delta\dot{\theta}_{VR}$, $x_9 = \hat{F}_{LR}$, $x_{10} = \hat{F}_{MR}$, $x_{11} = \hat{F}_{SR}$, $x_{12} = \hat{F}_{IR}$,

and following these notations, we can rewrite equations (35), (36), (37), (38), (47), (48), (49), (50), (51), (52),(53), and (54),

$$K_{SE}(x_3 - x_1) = \frac{x_9K_{SE}}{K_{SE}+K_{LT}} - \frac{x_1K_{SE}}{K_{SE}+K_{LT}} - \hat{B}_{AG}\dot{x}_3 \quad \textbf{(55)}$$

$$\begin{aligned} K_{SE}(x_1 - x_4 - x_6 + x_5) \\ = \frac{(x_{10}+x_{12}+x_{11})}{K_{SE}+K_{LT}}K_{SE} + \frac{x_1K_{SE}}{K_{SE}+K_{LT}} \\ + \hat{B}_{ANT}\dot{x}_4 + \hat{B}_{ANT}\dot{x}_6 - \hat{B}_{AG}\dot{x}_5 \end{aligned} \quad \textbf{(56)}$$

$$J\,\dot{x}_7 = K_{SE}(x_3 - x_1) - K_{SE}(x_1 - x_4 - x_6 + x_5) - K_px_1 - B_px_7 \quad \textbf{(57)}$$

$$K_{SE}(x_5 - x_2) = \frac{x_{11}K_{SE}}{K_{SE}+K_{LT}} - \frac{x_2K_{SE}}{K_{SE}+K_{LT}} - \hat{B}_{AG}\dot{x}_5 \quad \textbf{(58)}$$

$$\begin{aligned} K_{SE}(x_2 - x_6 - x_4 + x_3) \\ = \frac{(x_{10}+x_{12}+x_9)}{K_{SE}+K_{LT}}K_{SE} + \frac{x_2K_{SE}}{K_{SE}+K_{LT}} \\ + \hat{B}_{ANT}\dot{x}_4 + \hat{B}_{ANT}\dot{x}_6 - \hat{B}_{AG}\dot{x}_3 \end{aligned} \quad \textbf{(59)}$$

$$J\dot{x}_8 = K_{SE}(x_5 - x_2) - K_{SE}(x_2 - x_6 - x_4 + x_3) - K_px_2 - B_px_8 \quad \textbf{(60)}$$

$$\dot{x}_9 = \frac{N_{LR} - x_9}{\tau_{LR}} \quad \textbf{(61)}$$

$$\dot{x}_{10} = \frac{N_{MR} - x_{10}}{\tau_{MR}} \quad \textbf{(62)}$$

$$\dot{x}_{11} = \frac{N_{SR} - x_{11}}{\tau_{SR}} \quad \textbf{(63)}$$

$$\dot{x}_{12} = \frac{N_{IR} - x_{12}}{\tau_{IR}} \quad \textbf{(64)}$$

$$\dot{x}_1 = x_7 \quad \textbf{(65)}$$

$$\dot{x}_2 = x_8 \quad \textbf{(66)}$$

These twelve differential equations can be presented in a matrix form,

$$\dot{x} = Ax + u \quad \textbf{(67)}$$

Where $x, \dot{x}, u$ are 1x12 vectors, A is a square 12x12 matrix. Equation (67), completely describes the Oculomotor Plant mechanical model during saccades of the Right Upward eye movement. The form of the equation (67) gives us the opportunity to present oculomotor plant model in the Kalman filter form , therefore providing us with an ability to incorporate the 2DOPMM in a real-time online system with direct eye gaze input as reliable and robust eye movement prediction tool, therefore providing compensation for detection/transmission delays [2], [3].

Once equation (67) is solved, individual forces T_{LR}, T_{MR}, T_{IR}, and T_{SR} can be computed using equations (32), (12), (14), and (16), respectively.

B. *Muscle Model Equations*

This section presents equations for extraocular muscle forces during all directions of the eye globe. These equations can be employed for extraocular muscle effort estimation.

With the use of Right Upward and Left Downward eye movements, we found feature properties in each eight equations of movements. These feature properties are used to build muscle model equations, which can be used to calculate eight equations in all four types of eye movements: Right Upward, Right Downward, Left Upward, and Left Downward. In order to calculate eight force equations for a particular eye movement, muscle model properties needs to be identified. In our model we found following model properties in each of the eye movement to successfully predict force equations [14].

Plane – Horizontal (HR) or Vertical (VR)
ForceDirection – Top (T), Bottom (B), Right (R), Left (L)
AgonistMuscles – Agonist Muscles of the eye movement.
AntagonistMuscles – Antagonist of the eye movement model.

Sign – For this research, following ForceDirection sign convention was used. ForceDirections (R), (T) as positive, and ForceDirections (L), (B) as negative.

Above properties can be easily extracted with the use of a muscle model diagram [14].

- Equations-(4), (20), (39), and (43), are used to formulate muscle model equation 1,

$$T_{plane\_ForceDirection\_MF} = \pm K_{SE}\left(\Delta\theta_{plane\_LT\_AgonistMuscle} - \Delta\theta_{plane}\right) \quad \textbf{(68)}$$

- Equations-(3), (19), (40), and (44), are used to formulate following muscle model equation 2.

$$T_{plane\_ForceDirection\_MF} = \pm \frac{\hat{F}_{AgonistMuscle}K_{SE}}{K_{SE}+K_{LT}} \mp \frac{\Delta\theta_{plane}K_{SE}}{K_{SE}+K_{LT}} \mp \hat{B}_{AG}\Delta\dot{\theta}_{plane\_LT\_AgonistMuscle} \quad \textbf{(69)}$$

- Equations-(18), (34), (41), and (45), are used to formulate muscle model equation 3.

$$T_{plane\_ForceDirection\_MF} = \pm K_{SE}\big(\Delta\theta_{plane} - \Delta\theta_{plane\_LT\_AntagonistMuslce} - \Delta\theta_{PerpendicularPlane\_LT\_AntagonistMuscle} + \Delta\theta_{PerpendicularPlane\_LT\_AgonistMuscle}\big) \quad \textbf{(70)}$$

- Equations-(17), (33), (42), and (46), are used to formulate muscle model equation 4.

$$T_{plane\_ForceDiretion\_MF} = \pm \frac{\left(\hat{F}_{AgonistMuscle} + \hat{F}_{AntagonistMuscle} + \hat{F}_{AntagonistMuscle}\right)}{K_{SE}+K_{LT}}K_{SE} \pm \frac{\Delta\theta_{plane}K_{SE}}{K_{SE}+K_{LT}} \mp \hat{B}_{AG}\Delta\dot{\theta}_{PerpendicularPlane\_LT\_AgonistMuscle} \pm \hat{B}_{ANT}\Delta\dot{\theta}_{plane\_LT\_AntagonistMuscle} \pm \hat{B}_{ANT}\Delta\dot{\theta}_{PerpendicularPlane\_LT\_AntagonistMuscle} \quad \textbf{(71)}$$

Equations (68), (69), (70), and (71), allow us to come to the general matrix form represented by Equation (67) for all directions of eye movements: Right Upward, Left Upward, Right Downward, Left Downward, therefore providing the trajectories of movement and the estimation of the individual muscle forces T_{LR}, T_{MR}, T_{IR}, T_{SR}.

VI. CONCLUSION

Eye mathematical modeling can be used to advance such fast growing areas of research as medicine, human computer interaction, and software usability. In this paper we have created a two dimensional mechanical model of the human eye that is capable of generating eye movement trajectories with both vertical and horizontal components during fast eye movements (saccades) given the coordinates of the onset point, the direction of movement and the value of the saccade amplitude.

The important contribution of the proposed model to the field of bio engineering is the ability to compute individual extraocular muscle forces during a saccade – something that have not been done before to the best of our knowledge.

Our model evolved from a one dimensional version which was successfully employed for eye movement prediction as a tool for delay compensation in Human Computer Interaction with direct eye-gaze input [2], [3] and suggested for the effort estimation for improving the usability of the graphical user interfaces [4]. Two dimensional version of the model proposed in this paper extents those capabilities to a two dimensional plane.

In our future work, we intend to provide experimental evaluations of our model in terms of produced trajectories of movement and estimated individual extraocular muscle forces.

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