Algebraic Reconstruction Technique for Ultrasound Transmission Tomography

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Abstract

Ultrasound transmission tomography is a potentially promising alternative to standard X-Ray imaging in medical diagnosis, especially in mammography. The reconstruction of the local attenuation coefficient from the measured signals can be formulated as a large overdetermined system of linear equations based on a simplified ultrasound transmission model. It can be solved by means of the Kaczmarz algebraic reconstruction technique. The algorithm successively iterates through the equations and computes the corrections of the initial solution estimates. Because the original version of the algorithm does not guarantee convergence to the optimum, an extended version of the method is used here. It has been shown previously to converge to the least-mean-squares optimum. Both the original and extended algorithms are strictly sequential since the computation in the particular iteration depends on the corrections from the previous step. To enable parallelization of the method, thus speeding up the computation, a partitioning scheme is proposed and analyzed. The sequential as well as the partitioning-scheme algorithms are tested on both synthetic and real radiofrequency data (acquired using an experimental tomograph).

1. Introduction

Ultrasound transmission tomography is a potentially promising alternative to standard X-Ray imaging in medical diagnosis, especially in mammography. This is mainly due to the non-ionizing character of ultrasound and high information content of the measured signals that could potentially result in high-resolution imaging.

The measurement setup is similar to the MRI and X-Ray computed tomography setup [1]. The imaged object (e.g. human breast), immersed in a water tank, is surrounded by transducers emitting and receiving ultrasound field from various directions.

Compared to MRI and X-Ray computed tomography, the ultrasound field and the image reconstruction algorithms are rather complex and computationally demanding. This is because the wavelength of ultrasound and the size of the imaged structures are comparable which causes diffraction and refraction. As a result, ultrasound transmission tomography is in the research state, so far still not applicable in practice. More robust image restoration methods and application of more accurate mathematical models of the ultrasound field are needed.

Here, the ultrasound transmission tomography setup is considered with the aim to image a map of ultrasound attenuation coefficient of the immersed object. This tissue parameter is closely related to the tissue type and its pathological state and, thus, is of high diagnostic value [2].

In the published methods [3, 4, 5], the principle is directly derived from computed tomography. The emitted ultrasound pulse is supposed to propagate along a narrow straight line. Attenuation along the beams are estimated and are arranged to projections. The problem of attenuation-coefficient image reconstruction is then formulated as the inverse Radon transform [1]. It is solved by means of filtered backprojection. The major artifacts of these methods are caused by refraction, phase cancellation (due to distorted phasefront of the received pulse or the non-normal incidence), varying sound speed and pulse detection problems.

An alternative to the filtered backprojection are the so called algebraic reconstruction techniques (ART) [1]. The problem of inverse Radon transform is formulated as the solution of an overdetermined set of linear equations. This approach is more general because is can also be used for non-straight beams (e.g. reflected and scattered beams), which is potentially a valuable additional source of useful information [6]. Furthermore, due to the iterative character of the ART approach, additional regularization can be applied. Thus, the ART reconstruction is the method of preference here.

Kaczmarz method of projections [1] is a well accepted ART method. It gives satisfactory results for problems with square matrices, whereas the results for the overdetermined

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systems are not optimal. Therefore, an *extended Kaczmarz method* has been proposed [7]. It converges to the optimal solution of the overdetermined system of linear equations.

One of the original contributions of the paper is the application of ART methods in ultrasound transmission tomography. The basic and extended Kaczmarz methods are applied and analyzed. As the processed equation sets are fairly large (for the current experimental ultrasound tomograph about 130000 equations with 2000 unknowns, in future setups in 3D potentially millions of equations have to be solved), the solving is very computationally demanding. To enable parallelization of the method, thus speeding up the computation, a partitioning scheme is proposed and analyzed.

The sequential as well as the partitioning-scheme algorithms are tested on both synthetic and real radiofrequency data acquired using an experimental tomograph.

2. Formulation of the Image Reconstruction Problem

The present ultrasound attenuation tomography approach is derived for a setup, where the imaged object, immersed in a water tank, is enclosed by a ring of transducers (Fig. 1). One transducer is in the emitting mode, while all other transducers record the received radiofrequency signals. Then, the next element is selected as emitting and all remaining transducers are recording, and so on until all transducers have been used as emitters.

The emitted pulse is an undirected beam in the tomographic plane. Thus, the pulse spreads as a spherical wave in this plane. In the direction normal to the tomographic plane the pulse is supposed to be narrow. Such transmitted fields can be approximately achieved by a transducer with a small cross-section in the tomographic plane and focused to this plane. The recorded radiofrequency signals are long enough to contain the directly transmitted signal and also the signal reflected and scattered from the imaged structures in the tomographic plane.

In this study, only the first pulse $s(t)$ of the radiofrequency signal $rf(t)$ is used in the computation (see Fig. 1). It corresponding to the directly transmitted wave.

Taking any combination of the sending and receiving elements, the amplitude spectrum of the first pulse signal is

$$|S(\omega)| = |S_0(\omega)| \cdot e^{-\beta d |\frac{\omega}{2\pi}|}, \tag{1}$$

where ω is frequency, d is the distance between the sending and receiving element and $S_0(\omega)$ is the spectrum of a pulse recorded with no object between the sending and receiving transducers (only water in the measurement tank). It describes the electrical signal applied to the input of the sending transducer and the electroacoustical transfer functions of the transducers. β is the mean attenuation coefficient of the tissues along the direct propagation path l between the transducers.

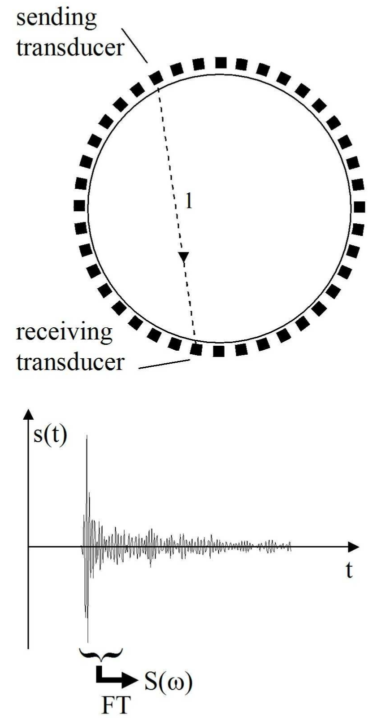

Figure 1.

Having a discretized attenuation-coefficient map in the tomographic plane, the mean attenuation coefficient β can be expressed by the local attenuation coefficients of each pixel β_i lying on the direct propagation path l:

$$\beta d = \sum_{i \in l} \beta_i d_i, \tag{2}$$

where d_i is the length of the i-th pixel along the path l. An overdetermined set of linear equations can be formulated based on Eq. 2 as

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}. \tag{3}$$

Each equation corresponds to one combination of the sending and receiving transducers. The column vector of unknowns $\mathbf{x}$ consists of the local attenuation coefficients β_m of all pixels located inside the ring of transducers. The matrix $\mathbf{A}$ consists of the pixel lengths $d_{m,n}$ along the corresponding propagation paths (nonzero for pixels on the path). The right-side column vector $\mathbf{b}$ consists of the attenuation terms βd for each propagation path. It can be estimated by several methods [3]. Here, the power-ratio method is used because it gave the most reliable results in our setup. It can be computed from the discretized spectra $S(k)$ and $S_0(k)$ (corresponding to the spectra $S(\omega)$ and $S_0(\omega)$) as

$$\beta d = -\frac{\pi}{\omega_0} ln \frac{\sum_k |S(k)|^2}{\sum_k |S_0(k)|^2}. \quad (4)$$

ω_0 stands for the center frequency of the transducers.

3. Solving the Overdetermined System of Linear Equations

In this section, the Kaczmarz method together with its modification proposed in [7] are briefly described. Then the proposed partitioning scheme adopted for the purpose of the parallelization is explained.

3.1. Kaczmarz Method

The Kaczmarz method [1] is based on the *method of projections* which naturally emerges from the physical character of the problem.

Let's assume we have a system of M equations and N variables describing the image. The bitmap image represented by a vector of N values can by considered as a single point in the N-dimensional space. Then, each equation of the system is regarded as a hyperplane. Providing that a unique solution exists, it is represented by the intersection of the hyperplanes. Using such a geometrical interpretation, the Kaczmarz method works as follows: starting in an arbitrary initial estimation p^0, the initial point is projected by perpendicular projection on the first hyperplane, giving the estimation point p^1. The process of projecting the actual estimation p^i on the $(i+1)$-th hyperplane is iteratively repeated. After reaching the M-th equation, the process iterates through the equations again, until the intersection is achieved.

For the sake of clarity we call the computation of the particular projection on the hyperplane (corresponding to processing of one equation) as an *inner iteration* and the repeated passing through the whole equation system as an *outer iteration*.

Apparently, the method does not require the system to be processed as a whole. The equations are processed sequentially. In the i-th inner iteration, only the result of the $(i-1)$-th inner iteration is needed. The method is suitable for large systems which cannot be fitted into the computer memory.

In the case of a square system of linear equations when $M = N$ the method computes the exact solution. However, we are interested in the overdetermined system of linear equations ($M >> N$) when there is no unique solution. In this case, the Kaczmarz method works as described above, but instead of reaching the intersection point, it oscillates in the neighborhood of the intersections of the hyperplanes. In this case, the optimal solution (as shown in the following sections) is usually not reached.

3.2. Kaczmarz Extended Algorithm

To assure convergence to the optimal solution also for inconsistent problems (overdetermined systems with the presence of noise), the extended Kaczmarz method has been developed [7]. The outer iteration consists of two phases. In the first phase, the right-hand side vector of the equation set is modified. Then, the second phase takes place in the same way as in the original Kaczmarz method, i.e. iterating through all the rows of the system, but using the modified right-hand side vector.

The first phase is analogous to the original Kaczmarz algorithm, with the difference that the vector of unknowns is fixed and the right-hand side vector is updated instead. Furthermore, the system matrix is processed column-by-column instead of row-by-row. Considering the i-th outer iteration, the first phase is performed as follows: the computation is started with an initial point q^0 in an M-dimensional space, which is either the initial right-hand side vector (for $i = 1$) or the right-hand side vector updated in the previous outer iteration (for $i > 1$). The columns of the system matrix are understood as hyperplanes in the M-dimensional space. In the first inner iteration, the initial point q^0 is projected on the first hyperplane represented by the first column of the matrix. This results in the updated point q^1. The process is iteratively repeated — in the j-th inner iteration, the actual point $q^{(j-1)}$ is projected on the j-th column of the matrix.

After passing through all N columns of the matrix, the point q^N is the corrected right-hand side vector used in the second phase as well as the initial point for the first phase in the following outer iteration.

The process of correcting the right-hand side vector and computation of the new estimation of the solution is repeatedly iterated until the required precision is achieved.

3.3. Partitioning Scheme

In the described image reconstruction, large equation systems containing potentially millions of equations are processed. Thus, parallelization of the methods is of particular interest. However, in both the original and extended versions, the computation in each inner iteration depends on the result of the previous one. Therefore, a *partitioning scheme* is proposed, which allows a straightforward parallelization. It is described in the following for the case of the original Kaczmarz method (the partitioned computation of the modified right-hand vector in the extended Kaczmarz method is analogous).

Before the first outer iteration is started, the equation system is vertically partitioned into K blocks, each having M/K equations. Then, the starting point p_i^0 of each block B_i is set to the same initial value p^0 (a zero vector). Then, inside a block B_i, the corrections are iteratively computed using M/K orthogonal projections of the actual point onto the rows of the block as in the original Kaczmarz method. The i-th inner iteration inside the block uses the correction computed in the previous inner iteration inside the same block. After processing all the equations in all blocks, we get K corrections (one from each block) and apply all of them on the initial starting point p^0. If the obtained vector fulfils the precision requirement, the method is stopped, otherwise new outer iteration is started.

Apparently, this partitioning of the system results in a simple method for parallelization — each block can be processed on a different computational node and after all blocks have been processed, the corrections are applied and a new distributed outer iteration can be started. However, due to the partitioning the behavior of the method is substantially changed. In the following text, we present results of experimental measurements showing how the precision and convergence of the method are affected by the partitioning. No mathematical derivation has been performed so far.

4. Experimental Setup and Results

4.1. Experimental Data

Two data sets were used to test and evaluate the attenuation image reconstruction methods. Synthetic radiofrequency signals were generated for a simulated object using Eq. 1 including scattering (1000 scatterers, $S_0(\omega)$ modelled as a sine function with Gaussian envelope). The geometry of the modelled ultrasound tomograph was identical to the experimental tomograph described in the next section.

The second data set was acquired using an experimental ultrasound tomograph, developed at Forschungszentrum Karlsruhe [8]. It enabled recording of the transmitted, reflected and scattered signal within the tomographic plane. The system consisted of two 16-element transducers immersed in a water tank. One transducer was used as a source of the emitted ultrasound pulse. The second transducer was used as a receiver, whereas the radiofrequency signal of each element was recorded. Both transducers could be independently positioned on a ring (diameter 12 cm) to emulate a ring of transducers enclosing the imaged object in a water tank. The ring was divided into 100 equidistantly spaced positions. The full measurement of one object consisted of about 130 000 radiofrequency signals.

The testing data were recorded from an artificial object. It consisted of a cylindrical plastic holder filled with gelatine. Four cylindrical objects, plastic bags filled with different types of oil, were inserted in the gelatine.

4.2. Measure of Error and Stopping Criterium

The usual way of comparing the reference and the reconstructed image is the computation of the *image difference*. Having two matrices $a_{ref}(m,n)$ and $a_(m,n)$ representing the reference and the reconstructed image respectively, the image difference $\Delta(a,b)$ is computed as the Euclidean norm of the matrix difference:

$$\Delta(a,b) = \sqrt{\sum_m \sum_n [a_{ref}(m,n) - a(m,n)]^2}.$$

The image difference as the measure of the error between the original and the reconstructed image is applicable only in the case of synthetic data where the original reference image is known. Another error measure, used for both synthetic and measured data is described below.

Having an $M \times N$ overdetermined equation system $\mathbf{Ax} = \mathbf{b}$, the error measure is given by residual r based on the vector Euclidean norm:

$$r = ||\mathbf{r}||_2 = \sqrt{\sum_{m=1}^{M} |r(m)|^2},$$

where the vector $\mathbf{r}$ is computed as $\mathbf{r} = \mathbf{Ax} - \mathbf{b}$.

The absolute value of the residual is not suitable as the *stopping criterium* as for various systems of equations, the residuals of the acceptable solutions are different. Therefore, the difference of the two successive residuals $|r^{(i+1)} - r^{(i)}|$ is used instead. In all experiments the iterations were stopped if $|r^{(i+1)} - r^{(i)}| < 10^{-10}$.

The comparison of the residual and image difference for various numbers of partitions (Fig. 2) shows the similarity of both quantities. Therefore, in the following text, only residual is used as the error measure.

4.3. Accuracy and Convergence Speed of the Original and Extended Method

The graphs in Fig. 3 show the results of an experiment, where the solution of the system was computed for both the phantom as well as the synthetic data. The residual is given as a function of the number of partitions varying from one (no partitioning) up to 64.

The comparison shows that the extended Kaczmarz method converges to more accurate results compared to the original one. The residual value in the extended method is almost constant with increasing number of partitions, whereas for the original method it is improved

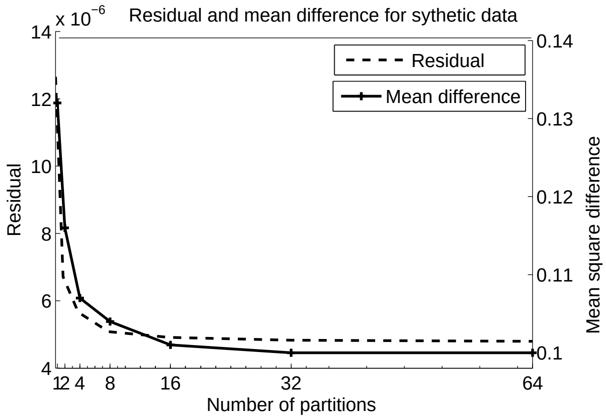

Figure 2. Comparison of the residual and the square image difference (measured using synthetic data)

with coarser partitioning (2,4 and 8 partitions). This phenomenon is also mentioned in [1]. Generally, the accuracy of the original method gets closer to the one of the extended version when partitioning is applied.

Fig. 4 shows the relation between the speed of convergence and the number of partitions. The measure of speed is expressed as the total number of equations that have to be solved within one partition during all outer iterations.

In both cases, the extended version of the method is computationally more expensive then the original one. Part of the reason is the fact that the number of inner iterations in each outer iteration is doubled for the extended method compared to the original version. This is due to the modification of the right-hand side vector (as described in section 3.2.

The speed of both algorithms improves with the number of partitions. It was observed that the convergence worsens with the increasing number of partitions, meaning need for more outer iterations. That is why the speed improvement due to the parallel computation of the equation blocks is not so high.

In general, the original version of the Kaczmarz method improves with coarser partitioning concerning both speed as well as the accuracy (with no important gain for more than 8 partitions). For the extended method, only the speed of the computation is improved (with no important speed gain for more than 8 partitions).

5. Conclusions

New possibilities of algebraic reconstruction techniques applied to attenuation image reconstruction in ultrasonic transmission tomography have been presented.

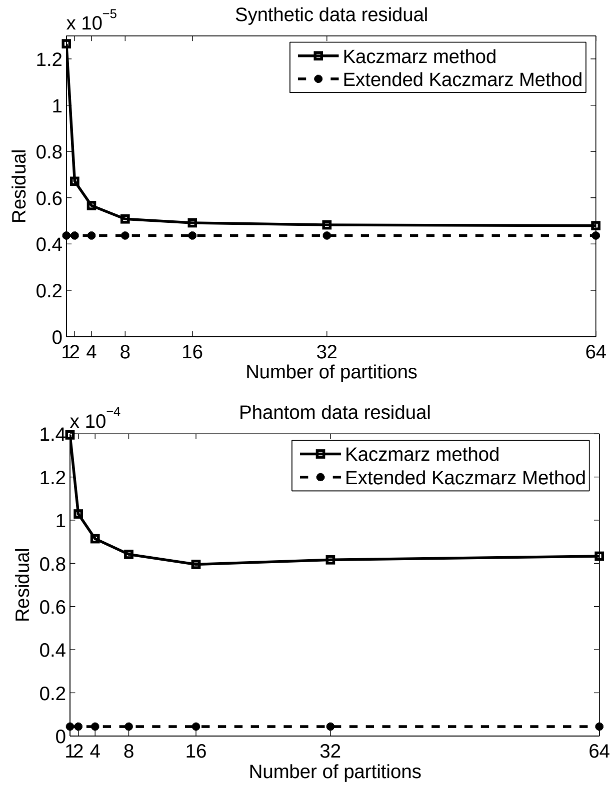

Figure 3. Evaluation of accuracy using residual (synthetic and phantom data)

The main contribution of the paper is the application of these techniques in ultrasound transmission tomography together with a new partitioning scheme.

The use of algebraic reconstruction techniques instead of standard filtered backprojection enables future incorporation of nonstraight propagation path of ultrasound beams (i.e. reflected and scattered wave) into the image reconstruction problem.

The partitioning scheme offers a straightforward parallelization of the algebraic reconstruction technique. It was adopted to both the original as well as the extended Kaczmarz methods. Both versions were compared with respect to the accuracy of the computed results and the speed of the computation. In both cases the partitioning leads to a considerable speed-up of the computation (up to 26× for the original method and 3× for the extended method). The partitioning is meaningful up to a certain number of partitions (about 8), for more partitions, the speed does not increase any more. The partitioning of the system also positively affects the accuracy of the original Kaczmarz method. On the other hand, the accuracy of the extended Kaczmarz method is not affected by the partitioning. The extended Kaczmarz

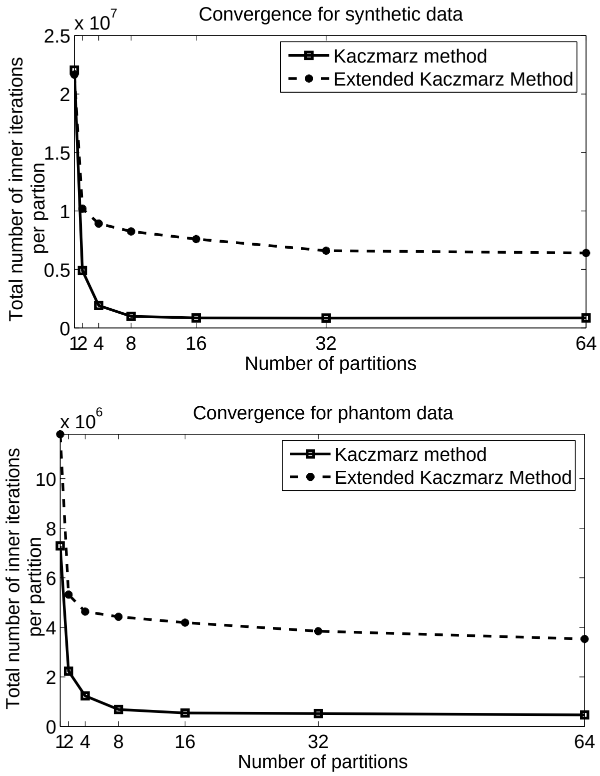

Figure 4. Speed of convergence (synthetic and phantom data)

method had superior accuracy over the original method.

In the future, the parallel version of the solver based on the partitioning scheme will be implemented for solving large overdetermined systems of linear equations (in order of millions), including the reflected and scattered field. Also some new methods of partitioning the system matrix based on the implicit properties of the modelled problem will be studied.

Acknowledgements

The project has been supported by the Czech Ministry of Education, Youth and Sports (Research Center DAR, proj. no. 1M6798555601) and the joint program of the German Academic Exchange Service and the Czech Academy of Science (grant. no. D-CZ 22/05-06). We are also grateful to MetaCentrum for offering the computational resources.

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