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A comparison of different choices for the regularization parameter in inverse electrocardiography models

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***Abstract*—Calculating the potentials on the heart's epicardial surface from the body surface potentials constitutes one form of inverse problems in electrocardiography (ECG). Since these problems are ill-posed, one approach is to use zero-order Tikhonov regularization, where the squared norms of both the residual and the solution are minimized, with a relative weight determined by the regularization parameter. In this paper, we used three different methods to choose the regularization parameter in the inverse solutions of ECG. The three methods include the L-curve, the generalized cross validation (GCV) and the discrepancy principle (DP). Among them, the GCV method has received less attention in solutions to ECG inverse problems than the other methods. Since the DP approach needs knowledge of norm of noises, we used a model function to estimate the noise. The performance of various methods was compared using a concentric sphere model and a real geometry heart-torso model with a distribution of current dipoles placed inside the heart model as the source. Gaussian measurement noises were added to the body surface potentials. The results show that the three methods all produce good inverse solutions with little noise; but, as the noise increases, the DP approach produces better results than the L-curve and GCV methods, particularly in the real geometry model. Both the GCV and L-curve methods perform well in low to medium noise situations.**

***Keywords*—Electrocardiography, inverse problem, L-curve, GCV, discrepancy principle**

I. Introduction

The problem of non-invasively computing the potentials on the heart's epicardial surface from the potentials measured on the body surface constitutes one type of electrocardiographic inverse problem [1]. Since the problem is ill-posed, small perturbations in the measured body surface potentials lead to large perturbations in the epicardial surface potentials. The inverse problem of ECG is a linear least-squares problem

$$\min \| \mathbf{A\Phi}_H - \mathbf{\Phi}_B \|_2 \quad (1)$$

where $\mathbf{\Phi}_\mathrm{B}$ is the column vector of m measured body surface potentials, which always have noise, $\mathbf{\Phi}_\mathrm{H}$ the desired solution of vector of n epicardial potentials, and A is the $\mathrm{m}\times\mathrm{n}$ transfer matrix relating the two vectors. The symbol $\|\cdot\|_2$ denotes the Euclidean norm of vector space. The common and well known approach used in the ill-posed problem is zero-order Tikhonov regularization, in which the solution is obtained by minimizing the following functional

$$f(\lambda) = \| \mathbf{A\Phi}_H - \mathbf{\Phi}_B \|_2^2 + \lambda \| \mathbf{\Phi}_H \|_2^2 \quad (2)$$

Here, the squared norms of both the residual and the solution are minimized together and the corresponding solution is

$$\mathbf{\Phi}_\mathrm{H} = (\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I})^{-1}\mathbf{A}^T\mathbf{\Phi}_B \quad (3)$$

where $\mathbf{I}$ is the $\mathrm{n}\times\mathrm{n}$ identity matrix and λ is the regularization parameter which controls the weight given to minimization of the side constraint relative to minimization of the residual norm. With different choices of λ, various solutions of $\mathbf{\Phi}_\mathrm{H}$ are obtained, and a good regularization parameter could yield a fair balance between the perturbation error and the regularization error in the solution.

Several methods for choosing the regularization parameter have been proposed [2, 3, 4]. If the norm of noise in $\mathbf{\Phi}_\mathrm{B}$ is known prior in some simulation study, discrepancy principle (DP) can be used to choose λ [5]. This approach has been investigated in some depth by Johnson [6]. Other methods that do not depend on an a priori knowledge of noise levels in $\mathbf{\Phi}_\mathrm{B}$ are zero-crossing [7], generalized cross-validation (GCV) [8] and L-curve [2, 9, 10], etc. The GCV method, however, has received little attention as an alternate method.

In this paper, we used the three methods: L-curve, GCV and DP, to select the regularization parameter. We estimate the norm of noise by using a model function in the DP method [13]. The object of the paper is to study the different effects of the three methods on the solutions and thus help us to choose the most suitable method in the inverse ECG problems.

II. Methodology

A. *The L-curve*, *GCV and discrepancy principle choices for λ*

The L-curve, which was popularized by Hansen [2], involves a plot, using the log-log scale, of the norm of the solution $\|\mathbf{\Phi}_\mathrm{H}\|$ on the ordinate, against the norm of the residual $\| \mathbf{A\Phi}_H - \mathbf{\Phi}_B \|_2$ on the abscissa, with λ as a parameter along the resulting curve. The curve is always in the form of an "L" and the λ at the corner of the "L" is taken as the solution λ_L. Hansen also points out the three conditions that ensure the existence of the corner [2].

The GCV is based on the philosophy that if an arbitrary element ϕ_{Bi} of $\mathbf{\Phi}_B$ is left out, then the corresponding regularized solution should well predict the observation. The GCV approach is to select the regularization parameter λ_G which minimizes the GCV function

$$G(\lambda) = \frac{1}{n} \| (\mathbf{I} - \mathbf{A}(\lambda))\mathbf{\Phi}_B \|^2 \Big/ \left[\frac{1}{n} Trace(\mathbf{I} - \mathbf{A}(\lambda)) \right]^2 \quad (4)$$

where

$$\mathbf{A}(\lambda) = \mathbf{A}(\mathbf{A}^T\mathbf{A} + n\lambda\mathbf{I})^{-1}\mathbf{A}^T \quad (5)$$

The minimization of the GCV function approximately minimizes the expected mean squared error of predictions of the transformed data $\mathbf{\Phi}_B$ with an estimated linear model [8].

The DP approach, which has received a considerable amount of attention in linear inverse problems [4, 11], is classified in Morozov principles including the damped Morozov principle [3,12,13]. In this paper, we only consider the former. This approach states that the regularization parameter λ_D should be chosen such that the error due to the regularization is equal to the error due to the observation data. That is, λ_D is chosen according to

$$\| \mathbf{A\Phi}_H - \mathbf{\Phi}_B \|_2^2 = \delta^2 \quad (6)$$

where δ is the observation error defined by

$$\| \mathbf{\Phi}_B^* - \mathbf{\Phi}_B \|_2^2 = \delta^2 \quad (7)$$

$\mathbf{\Phi}_B^*$ is the accurate potential which is not polluted by noise. We can see that the regularization solution satisfies:

$$as \ \| \mathbf{\Phi}_B^* - \mathbf{\Phi}_B \|_2^2 \to 0, \ \ \mathbf{\Phi}_H \to \mathbf{\Phi}_H^* \quad (8)$$

where $\mathbf{\Phi}_H^*$ is the accurate solution. This property is only present in the discrepancy principle. Therefore, if we can get δ as accurate as possible, the solution using the discrepancy principle can be more accurate than those of other approaches.

B. *Predictions of the observation errors*

The DP approach is based on the knowledge, or a good estimate of the observation error level δ. However, δ is often inaccessible, expensive to achieve or itself error-prone in practice, this is usually the case in ECG acquisition. In such situations, it can be helpful to utilize a heuristic approach to estimate δ. In the paper, we use the model function m(λ),which was proposed by Kunisch and Zou [13], to obtain an estimate of δ.

C. *Evaluation of the Solutions*

When the epicaridal potentials are known ahead of time, the solved epicardial potentials by inverse problem of ECG can be estimated by either the relative error (RE), namely

$$RE(\lambda) = \frac{\left\| \mathbf{\Phi}_H - \mathbf{\Phi}_H^* \right\|}{\left\| \mathbf{\Phi}_H^* \right\|} \quad (9)$$

or the correlation coefficient (CC), given by

$$CC(\lambda) = \frac{\sum_i^n \left[(\Phi_H)_i - \bar{\Phi}_H \right] \left[(\Phi_H^*)_i - \bar{\Phi}_H^* \right]}{\left\| \Phi_H - \bar{\Phi}_H \right\| \left\| \Phi_H^* - \bar{\Phi}_H^* \right\|} \quad (10)$$

where $\bar{\Phi}_H$ is the mean value.

D. *Simulation Protocol*

Two simulation protocols of increasing complexity were used to compare the three choices for regularization parameter λ. The first corresponds to using a concentric spheres model to represent the heart and body surfaces, with known analytical distribution induced by a limited number of current dipoles placed within the inner sphere. The radii of the spheres and the triangular boundary element are the same as that of Johnston [7]. The radii of inner and outer spheres are 4 cm and 10 cm, respectively. The spheres are discretized by uniform spherical coordinates. We derive a 192×122 transfer matrix **A** by the boundary element method [14]. At first, we use the analytical inner surface potential to compute the outer surface potential with the transfer matrix A. Then we added various signal to noise ratio (SNR) measure noises to the outer sphere potentials to get $\mathbf{\Phi}_B$ with noise, in order to test the three approaches.

The second simulation protocol is the realistic heart and torso geometry, which has been described by Lu and Xia [15]. The torso and epicardial surface were divided into 412 and 492 nodes, 820 and 944 triangles connecting these nodes, respectively. The transfer matrix is larger and more complex than that of concentric model. The body surface potentials are computed from the epicardial potentials with some radial dipoles located in the heart and Gaussian noise id added to mimic real dada.

III. RESULTS

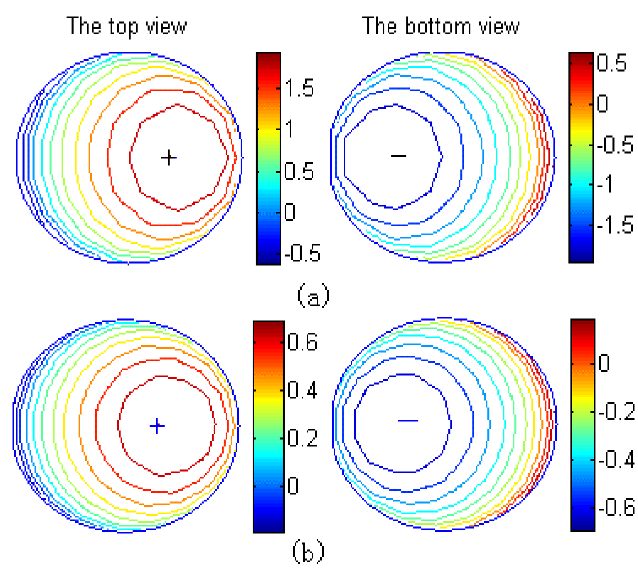

Figure 1. Analytically computed inner (a) and numerically computed outer (b) sphere potential distributions due to a single dipole. Each circle represents the projected view of one hemisphere and each line represents the equal potential value. The maximum and minimum of potentials are indicted as '+'and '-' in the picture.

A. *Concentric-Spheres Simulations*

Our initial simulation was made with the concentric spheres model. Fig.1 shows the analytically computed inner and the numerically computed outer sphere distribution due to a single dipole, direct from bottom to top with an angle 30°. Fig. 1(a) shows the analytically computed inner sphere distribution due to a single dipole, while Fig. 1(b) is the computed outer sphere distribution by boundary element method from the analytically computed inner sphere potentials. There are maximum and minimum potential in the inner and outer sphere as shown in the Fig.1. The RE of numerically computed outer sphere potentials is 0.08 and CC is 0.99, which means the boundary element method used in the paper performs well. Fig. 2 shows the inverse solutions obtained from the body surface potentials with the measurement noise by the three various choices for λ. The noise is simulated as the Gaussian white noise with the SNR 20:1.We also simulated other noise levels, the results are summarized in the table I.

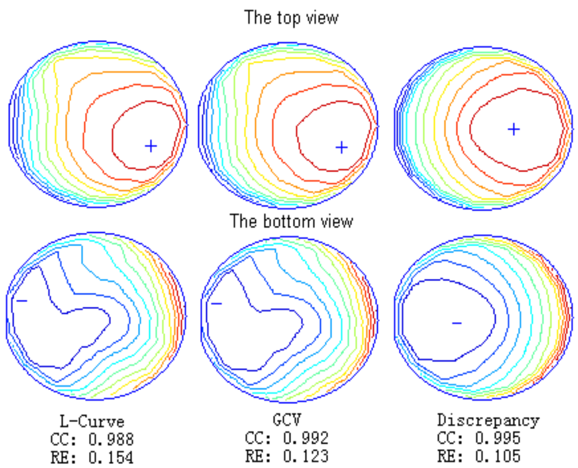

Figure 2. Inverse solutions of concentric spheres with measure noise (see text). From left to right, they are the solutions with three regularization parameter choices approaches: L-Curve, GCV and Discrepancy principle. The CC and RE are also indicated in the figure.

TABLE I Summary of inversion for two concentric spheres with one dipole within the inner sphere in different noise level.

Noised added (SNR)	Methods	CC	RE
50:1	L-Curve GCV DP	0.976 0.999 0.994	0.221 0.048 0.106
20:1	L-Curve GCV DP	0.988 0.992 0.995	0.154 0.123 0.105
10:1	L-Curve GCV DP	0.982 0.969 0.991	0.189 0.249 0.134
5:1	L-Curve GCV DP	0.976 0.952 0.992	0.252 0.310 0.129

B. *Simulations with Realistic Heart-Torso Geometries*

As mentioned above, the second set of simulations was done with the real geometries of heart and torso, which was reconstructed from CT images, and still uses a limited number of dipoles. We plot only the epicardial surface of ventricle. Fig. 3 is the distribution of epicardial and body surface potentials due to two radial dipoles, placed inside the realistic heart model. We perfomed the same simulation in the concentric spheres model. The inverse solutions from the body surface potentials with the SNR 50:1 Gaussian noise added are shown in Fig. 4. Table II summarizes the results of the entire second set of simulations.

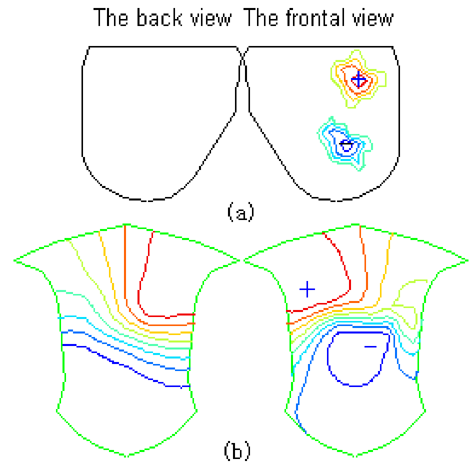

Figure 3. Epicardial and body surface potential distributions due to two radial dipoles, one pointing inwards and one outwards, placed inside a realistic heart model: (a) Epicardial potential distribution as plotted from the potential calculated at 492 nodes and (b) Body surface potential distribution calculated from the epicardial distribution of (a) using the boundary element method. Front (left) and back (right) views are shown.

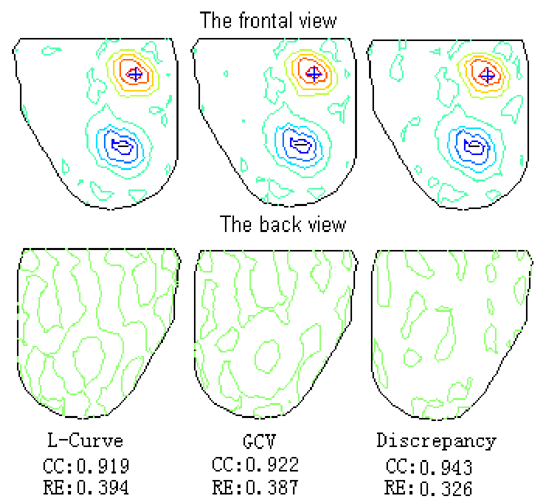

Figure 4. Inverse solutions with two radial dipole sources and real heart-torso models, adding 2% Gaussian noises in the body surface potentials.

TABLE II Summary of inversion for realistic heart-torso geometries with two radial dipoles within the heart.

Noised added (SNR)	Methods	CC	RE
50:1	L-Curve GCV DP	0.919 0.922 0.943	0.394 0.387 0.326
20:1	L-Curve GCV DP	0.850 0.850 0.876	0.527 0.530 0.412
10:1	L-Curve GCV DP	0.763 0.758 0.800	0.651 0.656 0.498

IV. Discussion And Conclusion

The L-curve, the generalized cross validation (GCV) and the discrepancy principle (DP) regularization parameter selection methods in the ECG inverse problem have been investigated and compared. The results show that all the three methods can achieve good inverse solutions with little noise. As the noise increases, the DP approach can get better results than the L-curve and the GCV, particulaly in the real geometry model. However, the DP needs the prior knowledge of noise. For practical reason, the L-curve and the GCV methods maybe more suitable for regularization in the ECG inverse problem. Since the GCV method has not been extensively investigated in the field of ECG inverse problems, we suggest that the GCV method warrants further investigation.

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