Proceedings of the 28th IEEE
EMBS Annual International Conference
New York City, USA, Aug 30-Sept 3, 2006

Comparing Unscented and Extended Kalman Filter Algorithms in the Rigid-Body Point-Based Registration

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Abstract— **Rigid registration is a crucial step in guidance system designed for neurosurgery, hip surgery, spine surgery and orthopaedic surgery. These systems often rely on point-based registration to determine the rigid transformation. The points used for registration (fiducial points) can be either extracted from the object being registered, or created by implanting fiducial markers on the object. The localized fiducial points are generally corrupted by the noise which is called fiducial (point) localization error. In this work, we present a new point-based registration algorithm based on the Unscented Kalman Filter (UKF) algorithm and compare it with the earlier proposed registration algorithm which is based on the Extended Kalman Filter (EKF) algorithm. By means of numerical simulations, it is shown that the UKF registration algorithm more accurately estimates the registration parameters than the EKF registration algorithm. In addition, in contrast with EKF, UKF computes the variance of the estimated registration parameters with the accuracy of at least second-order Taylor series expansion. The computed variances are valuable information that can be used to determine the accuracy of the registration at any desired target positions (target registration error). We utilize the estimated variance of the registration parameters to compute the distribution of target registration error (TRE) at a desired target location. A new formula for the distribution of TRE, based on the estimated variances, is derived, and it is shown that the computed distribution more accurately follows the real distribution that is generated by the numerical simulations, than the one obtained from the EKF registration algorithm.**

I. INTRODUCTION

The goal is to estimate the optimum transformation that aligns two data sets, generated from a rigid object in two spaces, in the presence of the point localization error. The error in the data sets is modelled by a random variable whose distribution is assumed to be the zero-mean isotropic Gaussian. This problem, also called the absolute orientation problem, has been shown to have a closed-form solution based on the block estimation algorithms when the data sets are corrupted by isotropic noise. Schönemann [1] published the first solution to that problem in 1966 which was based on the orthogonal procrustes. His solution was rediscovered independently by Golub and van Loan [2] in 1983. Also, Arun *et al.* [3] derived a closed-form solution for the absolute orientation problem based on the singular value decomposition in 1977. Horn proposed other closed-form solutions for that problem by using the unit Quaternions and the orthonormal matrices in 1977 [4] and 1988 [5], respectively. Finally, Walker [6], in 1991, came up with another solution for the absolute orientation problem by using the dual Quaternions. In 1995, Lorusso *et al.* [7] showed that these closed-form solutions were basically equivalent to each other and none of them was superior in all cases. However, all the algorithms utilize the block estimation methods to register two data sets. Pennec *et al.* [8], in 1997, used the Extended Kalman Filter (EKF) algorithm to present a sequential estimation method to register two data sets in the presence of isotropic/anisotropic Gaussian noise. However, this algorithm is accurate only to the first-order Taylor series expansion.

In this work, we propose to use the Unscented Kalman Filter (UKF) algorithm, instead of EKF, for registering two data sets because: 1) UKF avoids the derivation of jacobian matrices, and 2) UKF is valid for higher-order (at least second-order) estimation of the Taylor series expansion than the standard EKF algorithm with the same amount of computational complexity [9], [10]. We have implemented both UKF and EKF algorithms and performed several numerical simulations to verify superiority of the proposed UKF registration algorithm over EKF. It is illustrated that not only UKF converges faster to the optimum solution, but also it estimates the variance of the estimated transformation parameters more accurately than EKF. The estimated variances are then utilized to calculate the distribution of target registration error (TRE) at an arbitrary target position using the derived closed-form formula. We verify this distribution against Fitzpatrick's [11] which is based on the Monte Carlo simulation. It is shown that the derived TRE distribution much closely equates the one generated by the numerical simulations, comparing with the ones obtained from EKF and the approach proposed in [11]. The remainder of this paper is organized as follows: in Section II, the point-based rigid-body registration method, based on the EKF and UKF algorithms are presented. The distribution of TRE at an arbitrary target location is derived in Section III. Numerical analysis and simulation results to verify the proposed algorithms are come in Section IV. Finally, the conclusion and future work are discussed in Section V.

II. SYSTEM IDENTIFICATION AND METHODOLOGY

Here, we indicate the number of points in each space by N and dimension of the space containing the points by 3. The points in one space (moving data set) is shown by $\mathbf{U}$, as a $3\times N$ matrix whose columns correspond to the position vectors of points in that space. $\mathbf{Y}$, as a $3\times N$ matrix, represents the corresponding points in the second space (fixed data set). The ith columns of $\mathbf{U}$ and $\mathbf{Y}$ represent a pair of corresponding points in the two spaces. In addition,

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the both data sets are assumed to be perturbed by zero-mean isotropic Gaussian noise which models the point localization error in the data sets. As in [11], We make a simplifying assumption that the point localization error in the moving space $\mathbf{U}$ is negligible (identically zero) in comparison to the point localization error in the fixed data set $\mathbf{Y}$ by adding the variance of the noise in the moving data set (σ_U^2) to the one in the fixed data set (σ_Y^2). The goal is to determine a 3×3 rotation matrix $\mathbf{R}$ and a 3×1 translation vector $\mathbf{t}$ from the following non-linear observation model:

$$\mathbf{y}_{1:i} = \mathbf{R}_{(\theta_\mathbf{x},\theta_\mathbf{y},\theta_\mathbf{z})}\mathbf{u}_{1:i} + \mathbf{t}_{(t_x,t_y,t_z)} + \mathbf{n}_{1:i}, \qquad (1)$$

where, $\mathbf{y}_{1:i} = [\mathbf{y}_1^T, ..., \mathbf{y}_i^T]^T$, $\mathbf{u}_{1:i} = [\mathbf{u}_1^T, ..., \mathbf{u}_i^T]^T$, $\mathbf{n}_{1:i} = [\mathbf{n}_1^T, ..., \mathbf{n}_i^T]^T$ and $i = 1, ..., N$. $\mathbf{n}_i$ is the point localization error for point i which is assumed to be a zero mean Gaussian 3×1 random vector with covariance matrix $\mathbf{\Sigma}_n = (\sigma_U^2 + \sigma_Y^2)I_{3\times3}$. $\mathbf{t} = [t_x, t_y, t_z]^T$ is the translation vector along orthogonal coordinate frame's axes and θ_x, θ_y and θ_z are the rotational angles about the orthogonal coordinate frame's axes, respectively. In this work, we have used the Euler angles to represent the rotation matrix $\mathbf{R}$. We have accounted for this representation's singularity points in our simulations by limiting the range in which each rotation angle could change. However, the same analysis presented here is valid for other rotation matrix representations such as Quaternions and rotation around a helical axis. Let us define the state vector $\mathbf{x}$ to be a 6×1 vector including the registration parameters as, $\mathbf{x} = [t_x, t_y, t_z, \theta_x, \theta_y, \theta_z]^T = [\mathbf{x}_t^T, \mathbf{x}_\theta^T]^T$, where $\mathbf{x}_R$ and $\mathbf{x}_t$ is defined to be $[\theta_x, \theta_y, \theta_z]^T$ and $[t_x, t_y, t_z]^T$, respectively. Also, Let us assume that the state model is defined as, $\mathbf{x}_i = \mathbf{x}_{i-1} + \mathcal{N}(0, \mathbf{\Sigma}_Q)$, with the initial value and covariance matrix $\mathbf{x}_0$ and $\mathbf{P}_x^0$, respectively. $\mathcal{N}(0, \mathbf{\Sigma}_Q)$ is a zero-mean Gaussian random vector with a covariance matrix $\mathbf{\Sigma}_Q$. Since the observation model, (1), is a nonlinear function of the state vector $\mathbf{x}$, one may use either Extended Kalman Filter (EKF) or Unscented Kalman Filter (UKF) algorithms to estimate the optimum state vector $\mathbf{x}$, in the sense of minimum mean squared error. In what follows, we implement both the EKF and UKF algorithms and compare their performances.

A. EKF registration algorithm

1) Predict the state vector and its covariance matrix from the state model as:

$$\hat{\mathbf{x}}_i^- = \hat{\mathbf{x}}_{i-1}, \quad \mathbf{P}_{\hat{\mathbf{x}}_i^-} = \mathbf{P}_{\hat{\mathbf{x}}_{i-1}} + \mathbf{\Sigma}_Q. \qquad (2)$$

2) Append the ith point from the moving data set $\mathbf{U}$ to the already collected points from that data set, and estimate their corresponding points' positions in the fixed data set, $\mathbf{Y}$, by using the predicted state vector in step one as:

$$\hat{\mathbf{y}}_{1:i} = \mathbf{R}_{(\hat{\theta}_\mathbf{x},\hat{\theta}_\mathbf{y},\hat{\theta}_\mathbf{z})}\mathbf{u}_{1:i} + \mathbf{t}_{(\hat{\mathbf{t}}_\mathbf{x},\hat{\mathbf{t}}_\mathbf{y},\hat{\mathbf{t}}_\mathbf{z})}. \qquad (3)$$

3) Determine the Jacobian of the observation model at $\hat{\mathbf{x}}_i^-$ as follows:

$$\begin{aligned}\mathbf{J}_{y_i} &\equiv \frac{\partial \mathbf{y}_i}{\partial \mathbf{x}}|_{\mathbf{x}=\hat{\mathbf{x}}_i^-} = \frac{\partial}{\partial \mathbf{x}}(\mathbf{R}_{(\theta_\mathbf{x},\theta_\mathbf{y},\theta_\mathbf{z})}\mathbf{u}_i + \mathbf{t}_{(\mathbf{t_x},\mathbf{t_y},\mathbf{t_z})}) \\ &= [\mathbf{I}_{3\times3}, \frac{\partial \mathbf{R}}{\partial \theta_\mathbf{x}}\mathbf{u}_i, \frac{\partial \mathbf{R}}{\partial \theta_\mathbf{y}}\mathbf{u}_i, \frac{\partial \mathbf{R}}{\partial \theta_\mathbf{z}}\mathbf{u}_i]_{3\times6}, \qquad (4)\end{aligned}$$

4) Compute the error in the estimated corresponding points in the data set $\mathbf{Y}$, $(\mathbf{y}_{1:i} - \hat{\mathbf{y}}_{1:i})$, to update the state vector and its covariance matrix as follows:

$$\hat{\mathbf{x}}^i = \hat{\mathbf{x}}_i^- + \mathbf{K}_{1:i}(\mathbf{y}_{1:i} - \hat{\mathbf{y}}_{1:i}), \qquad (5)$$
$$\mathbf{P}_{\hat{\mathbf{x}}_i} = \mathbf{P}_{\hat{\mathbf{x}}_i^-} - \mathbf{K}_{1:i}E[\mathbf{y}_{1:i}\mathbf{y}_{1:i}^T]\mathbf{K}_{1:i}^T, \qquad (6)$$

where $\mathbf{K}_{1:i} = \mathbf{P}_{\hat{\mathbf{x}}_i^-}/(\mathbf{\Sigma}_n + \mathbf{J}_{y_{1:i}}\mathbf{P}_{\hat{\mathbf{x}}_i^-}\mathbf{J}_{y_{1:i}}^T)$ is called the Kalman gain. This algorithm is iterated through all the points in the moving data set $\mathbf{U}$ to estimate the optimal state vector $\mathbf{x}$. It should be noted that the EKF registration algorithm is accurate up to the first-order Taylor series expansion since it does not include the higher order nonlinearity terms of the observation model in calculations.

B. UKF registration algorithm

1) Predict the state vector and its covariance matrix from the state model as:

$$\hat{\mathbf{x}}_i^- = \hat{\mathbf{x}}_{i-1}, \quad \mathbf{P}_{\hat{\mathbf{x}}_i^-} = \mathbf{P}_{\hat{\mathbf{x}}_{i-1}} + \mathbf{\Sigma}_Q. \qquad (7)$$

2) Append the ith point from the moving data set $\mathbf{U}$ to the already collected points from that data set, and estimate their corresponding points' positions in the fixed data set, $\mathbf{Y}$, by using the predicted state vector in step one as:

$$\hat{\mathbf{y}}_{1:i} = \mathbf{R}_{(\hat{\theta}_\mathbf{x},\hat{\theta}_\mathbf{y},\hat{\theta}_\mathbf{z})}\mathbf{u}_{1:i} + \mathbf{t}_{(\hat{\mathbf{t}}_\mathbf{x},\hat{\mathbf{t}}_\mathbf{y},\hat{\mathbf{t}}_\mathbf{z})}. \qquad (8)$$

3) Compute the error in the estimated corresponding points in the data set $\mathbf{Y}$, $(\mathbf{y}_{1:i} - \hat{\mathbf{y}}_{1:i})$, to update the state vector and its covariance matrix as follows:

$$\begin{aligned}\hat{\mathbf{x}}^i &= \hat{\mathbf{x}}_i^- + \mathbf{K}_{1:i}(\mathbf{y}_{1:i} - \hat{\mathbf{y}}_{1:i}), \qquad (9)\\ \mathbf{P}_{\hat{\mathbf{x}}_i} &= \mathbf{P}_{\hat{\mathbf{x}}_i^-} - \mathbf{K}_{1:i}E[\mathbf{y}_{1:i}\mathbf{y}_{1:i}^T]\mathbf{K}_{1:i}^T, \qquad (10)\end{aligned}$$

where $\mathbf{K}_{1:i} = E[\mathbf{x}_i\mathbf{y}_{1:i}^T]/E[\mathbf{y}_{1:i}\mathbf{y}_{1:i}^T]$ is the Kalman gain. Since the observation model is nonlinear, instead of making that linear, the Unscented Transform (UT) [12] is used to compute $E[\mathbf{x}_i\mathbf{y}_{1:i}^T]$ and $E[\mathbf{y}_{1:i}\mathbf{y}_{1:i}^T]$, respectively. It has been shown that the UT can estimate the first and second moments of any nonlinearities with the accuracy of at least second-order Taylor series expansion [12]. This algorithm is iterated through all the points in the moving data set $\mathbf{U}$ to estimate the optimal state vector $\mathbf{x}$.

III. Derivation of TRE distribution

Let's assume that the registration is performed between the moving and the fixed data sets by using either the UKF or EKF registration algorithms. In addition, let $\mathbf{r}$ be a target point, other than the ones used in the registration algorithm, for which we would like to compute TRE$_r$. The registration error at the target point $\mathbf{r}$ can be written as:

$$\begin{aligned}\mathbf{e}(\mathbf{r}) &= \mathbf{R}_{(\hat{\mathbf{x}}_\mathbf{R})}\mathbf{r} + \hat{\mathbf{t}}_{(\hat{\mathbf{x}}_\mathbf{t})} - \mathbf{R}\mathbf{r} - \mathbf{t} \qquad (11)\\ &= (\mathbf{R}_{(\hat{\mathbf{x}}_\mathbf{R})} - \mathbf{R})\mathbf{r} + (\mathbf{t}_{(\hat{\mathbf{x}}_\mathbf{t})} - \mathbf{t}) = \Delta\mathbf{R}\mathbf{r} + \Delta\mathbf{t},\end{aligned}$$

where TRE$_r$ in terms of $\mathbf{e}(\mathbf{r})$ has been defined as $\sqrt{||\mathbf{e}(\mathbf{r})||^2}$. If the registration algorithm converges to the true solution, $\Delta\mathbf{t}$ can be approximated by a zero-mean Gaussian random vector with covariance matrix $\mathbf{\Sigma}_{\Delta t}$ which can be obtained from $\mathbf{P}_{\hat{\mathbf{x}}}$ as follows:

$$\mathbf{P}_{\hat{\mathbf{x}}} = E[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T] = \begin{pmatrix} \mathbf{\Sigma}_{\Delta t} & \mathbf{\Sigma}_{\Delta\theta t} \\ \mathbf{\Sigma}_{\Delta\theta t} & \mathbf{\Sigma}_{\Delta\theta} \end{pmatrix}_{6\times 6}. \quad (12)$$

If the registration algorithm converges, and therefore $E[\Delta\mathbf{R}] = 0$, $\Delta\mathbf{Rr}$ can be also approximated by a zero-mean random vector with covariance matrix $\mathbf{\Sigma}_{\Delta Rr}$. $\mathbf{\Sigma}_{\Delta Rr} = E[(\Delta\mathbf{Rr})(\Delta\mathbf{Rr})^T]$ can be estimated by using the Unscented Transform [12], since $(\Delta\mathbf{Rr})$ is a nonlinear function of the Gaussian random vector $\mathbf{x}_\theta$. $\mathbf{x}_\theta$ has the Gaussian distribution with mean $\hat{\mathbf{x}}_\theta$ and covariance matrix $\mathbf{\Sigma}_\theta$ which are determined from (6) or (10) and (12), respectively. Here, we approximate the distribution of the error vector $\mathbf{e}$ by a zero-mean Gaussian distribution with covariance matrix $E[\mathbf{ee}^T]$ which can be determined as follows:

$$E[\mathbf{ee}^T]{=}E[(\Delta\mathbf{Rr})(\Delta\mathbf{Rr})^T]{+}2E[(\Delta\mathbf{Rr})\Delta\mathbf{t}^T]{+}E[\Delta\mathbf{t}\Delta\mathbf{t}^T].$$

$\mathbf{\Sigma}_{(\Delta Rr\Delta t)} = E[(\Delta\mathbf{Rr})\Delta\mathbf{t}^T]$ is the cross correlation matrix between $\Delta\mathbf{Rr}$ and $\Delta\mathbf{t}$, that can be computed by using the Unscented Transform and propagating the estimated state vector $\hat{\mathbf{x}}$ and its covariance matrix $\mathbf{P}_{\hat{\mathbf{x}}}$ through $(\Delta\mathbf{Rr}\Delta\mathbf{t}^T)$. Since TRE_r is a distance error and distance is invariant to rotations, the error vector $\mathbf{e}$ can be arbitrary rotated in a space while the distribution of TRE_r remains fixed. Thus, one may rotate $\mathbf{e}$ such that its covariance matrix, $\mathbf{\Sigma}_e$, becomes diagonal, and therefore, its components become independent zero-mean Gaussian random variables. Let's define the rotated error vector $\mathbf{e}$ as $[e_1, e_2, e_3]^T$, where e_1, e_2 and e_3 are zero-mean Gaussian random variables with covariance matrix $\mathbf{\Sigma}_e = diag(\lambda_1, \lambda_2, \lambda_3)$. λ_1, λ_2, and λ_3 are variances of e_1, e_2, and e_3, respectively. One can write TRE_r in terms of the rotated error vector $\mathbf{e}$ as $TRE_r = \sqrt{e_1^2 + e_2^2 + e_3^2}$. We first determine the probability distribution function (pdf) of TRE_r^2 from which the pdf of TRE_r can be easily computed. It is shown that TRE_r^2 is the summation of three squared independent zero-mean Gaussian random variables with variances of λ_1, λ_2 and λ_3. It means that the pdf of TRE_r^2 is the convolution of the pdf of each squared Gaussian random variable as follows:

$$\begin{aligned} pdf_{(TRE_r^2)}(x) &= pdf_{(e_1^2)}(x) * pdf_{(e_2^2)}(x) * pdf_{(e_3^2)}(x) \\ &= pdf_{(e_1^2+e_2^2)}(x) * pdf_{(e_3^2)}(x), \end{aligned} \quad (13)$$

where $*$ is the convolution operator and $pdf_{(e_1^2+e_2^2)}(x)$ is defined to be $pdf_{(e_1^2)}(x) * pdf_{(e_2^2)}(x)$. To do the convolutions, first the pdf of a squared Gaussian random variable should be calculated. It is easy to show that if e_i has a Gaussian distribution with mean zero and variance λ_i, the pdf of e_i^2 is $\frac{1}{\sqrt{2\pi x\lambda_i}}\exp(-\frac{x}{2\lambda_i})$ for x greater than or equal to zero. Consequently, $pdf_{(e_1^2+e_2^2)}(x)$ can be computed as:

$$pdf_{(e_1^2+e_2^2)}(x) = \frac{\exp(-\frac{x}{2\lambda_1})}{\sqrt{2\pi x\lambda_1}} * \frac{\exp(-\frac{x}{2\lambda_2})}{\sqrt{2\pi x\lambda_2}}, \quad x \geq 0. \quad (14)$$

Equation (14) can be simplified to:

$$pdf_{(e_1^2+e_2^2)}(x) = \frac{\exp[-\frac{x}{4}(\frac{1}{\lambda_1}+\frac{1}{\lambda_2})]}{2\sqrt{\lambda_1\lambda_2}} I_0(\frac{x}{4}(\frac{1}{\lambda_1}-\frac{1}{\lambda_2})), \quad (15)$$

where $I_0(x) = \frac{1}{\pi}\int_0^\pi exp(x\cos\phi)\,d\phi$. Finally, the pdf of TRE_r^2 can be obtained by using (13) as:

$$pdf_{(TRE_r^2)}(x) = \frac{\exp(-\frac{x}{2\lambda_3})}{2\sqrt{2\pi\lambda_1\lambda_2\lambda_3}} \times \int_0^x \frac{\exp(\frac{-\lambda}{4}[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{2}{\lambda_3}])}{\sqrt{(x-\lambda)}} I_0(\frac{\lambda}{4}[\frac{1}{\lambda_1}-\frac{1}{\lambda_2}])d\lambda. \quad (16)$$

Since $pdf_{(TRE_{(r)})}(x) = 2x \times pdf_{(TRE_{(r)}^2)}(x^2)$, the pdf of TRE_r can be derived as:

$$pdf_{(TRE_{(r)})}(x) = \frac{x\exp(-\frac{x^2}{2\lambda_3})}{\sqrt{2\pi\lambda_1\lambda_2\lambda_3}} \times \int_0^{x^2} \frac{\exp(\frac{-\lambda}{4}[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}-\frac{2}{\lambda_3}])}{\sqrt{(x^2-\lambda)}} I_0(\frac{\lambda}{4}[\frac{1}{\lambda_1}-\frac{1}{\lambda_2}])\,d\lambda. \quad (17)$$

IV. Simulation Results

In order to verify our derivations, several simulations are performed. In the first simulation, we choose N, the number of points in data sets to be 30 and 150, respectively. We generate the moving data set $\mathbf{U}$ by drawing N points uniformly within a cube in the range of ±250mm. The fixed data set $\mathbf{Y}$ is generated by applying a random transformation on the moving data set. The random transformation is produced by uniformly drawing the rotational and translational parameters along all axes within the ranges of $\pm 10^o$ and ±10mm, respectively. Also, a zero-mean Gaussian random noise with variance 6mm^2, $\mathcal{N}(0, 6mm^2)$, is added to the fixed data set to model the point localization error in the both data sets. Then, the UKF and EKF registration algorithms are employed to register the moving data set to the fixed one. To draw the error histogram of the estimated registration parameters, this procedure is repeated for 1000 trials. The error histogram of the estimated transformation parameters using the UKF and EKF registration algorithms for both cases where $N = 30$ and $N = 150$ are shown in Tables I and II, respectively. By looking at the Tables, one can conclude that the UKF registration algorithm needs less points than EKF to converge to the true solution. The performance of UKF and EKF for the translational parameters are the same since those parameters are the linear terms in the observation model. However, the problem arises in the EKF registration algorithm for the rotational parameters that are the nonlinear terms in the observation model. The UKF registration algorithm works better in estimation of the rotational parameters because it estimates the first and second moments of any nonlinearities with the accuracy of at least second-order Taylor series expansion by using the Unscented Transform; however EKF only considers the first-order term of the Taylor series expansion of any nonlinearities. It is seen that by increasing the number of registration points from 30 to 150, the EKF registration algorithm still can not reach to the performance of UKF. We have also observed that the EKF registration algorithm requires more than 200 points to get the same performance as that of UKF using 30 registration points. In the second simulation, we verify the proposed formula derived for the distribution of TRE

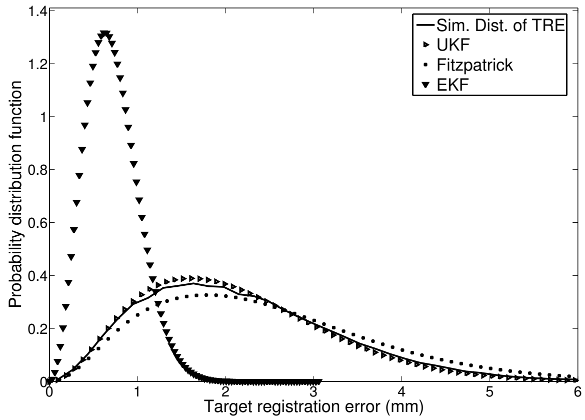

Fig. 1. Distribution of target registration error using different algorithms.

in Section III by comparing UKF, EKF and Fitzpatrick's algorithms, that compute the TRE distribution, with the one generated by the numerical simulations (considered as the gold standard). To do so, the moving data set $\mathbf{U}$ is generated by drawing 30 three-dimensional points uniformly within the ranges of $\pm$10mm (for the $\mathbf{x}$ components), $\pm$30mm (for the $\mathbf{y}$ components), and $\pm$60mm (for the $\mathbf{z}$ components), respectively. Also, one target position is selected randomly from a cube in the range of $\pm$100mm. In order to generate the fixed data set, $\mathbf{Y}$, we perturbed the data set $\mathbf{U}$ by a zero-mean Gaussian random noise with variance 6mm^2. In this way we produced the same model as that used by Fitzpatrick [11]. The moving data set $\mathbf{U}$ is then registered to the fixed one, $\mathbf{Y}$, and the TRE is measured at the target point by using (11). The distribution of TRE is estimated by repetitions of perturbation and registration steps for 100,000 times using random initial conditions. We next employed the UKF and EKF registration algorithms to register these two data sets to compute the variance of the estimated transformation parameters. Equation (17) is then used to estimate the distribution of TRE at the target position. Fig. 1 shows the estimated distribution of TRE at the selected target point using the UKF, EKF, Fitzpatrick's algorithms and the one obtained from the numerical simulations. As shown, the UKF registration algorithm much closely follows the real distribution, generated by the numerical simulations, since it estimates the variance of the transformation parameters with higher accuracy than the other implemented methods.

V. CONCLUSIONS AND FUTURE WORKS

The UKF and EKF registration algorithms are implemented and their performances are compared. It is shown that UKF works better than the EKF registration algorithm when the observation model is nonlinear in terms of the state vector. Additionally, the variance of the estimated transformation parameters is utilized to compute the distribution of TRE at a target position. A closed-form formula for computing the distribution of TRE, based on the estimated variances, is derived. It is observed that the UKF registration algorithm can closely estimate the distribution of TRE in comparing with the other methods because its estimation has the accuracy of at least second-order Taylor series expansion.

TABLE I

MEAN AND VARIANCE OF ERROR OF THE ESTIMATED TRANSFORMATION PARAMETERS ESTIMATED BY THE UKF AND EKF REGISTRATION ALGORITHMS WHEN $N = 30$.

Estimation Error	UKF		EKF	
	$Mean$	$Var.$	$Mean$	$Var.$
$\hat{t}_x - t_x(mm)$	-0.01	0.19	-0.01	0.19
$\hat{t}_y - t_y(mm)$	0.001	0.19	0.001	0.19
$\hat{t}_z - t_z(mm)$	-0.02	0.19	-0.02	0.19
$\hat{\theta}_x - \theta_x(deg)$	-0.005	0.02	0.02	12.6
$\hat{\theta}_y - \theta_y(deg)$	-0.0001	0.02	-0.06	12.6
$\hat{\theta}_z - \theta_z(deg)$	0.002	0.02	0.01	12.8

TABLE II

MEAN AND VARIANCE OF ERROR OF THE ESTIMATED TRANSFORMATION PARAMETERS ESTIMATED BY THE UKF AND EKF REGISTRATION ALGORITHMS WHEN $N = 150$.

Estimation Error	UKF		EKF	
	$Mean$	$Var.$	$Mean$	$Var.$
$\hat{t}_x - t_x(mm)$	-0.01	0.04	-0.01	0.04
$\hat{t}_y - t_y(mm)$	0.004	0.04	0.004	0.04
$\hat{t}_z - t_z(mm)$	-0.01	0.04	-0.01	0.04
$\hat{\theta}_x - \theta_x(deg)$	0.003	0.003	-0.009	0.2
$\hat{\theta}_y - \theta_y(deg)$	0.001	0.003	0.01	0.2
$\hat{\theta}_z - \theta_z(deg)$	0.003	0.003	-0.02	0.2

As a future work, we would like to increase the UKF registration algorithm's performance by using the higher-order Unscented Transform, instead of Unscented Transform, which estimates more moments of any nonlinearities with higher accuracy. Also it is desired to modify the UKF registration algorithm such that the point localization error can have any kind of distributions other than the Gaussian distribution which is considered in this paper.

REFERENCES

[1] P. H. Schönemann, A generalized solution of the orthogonal procrustes problem, *Psychometrika J.*, vol. 31, 1966, pp 1-10.

[2] G. Golub, and C. van Loan, *Matrix computations*, John Hopkins Univ. Press, Baltimore, MD; 1983.

[3] K. S. Arun, and T. S. Huang, and S. D. Blostein, Least squares fitting of two 3D point sets, *IEEE Trans. on Pattern Analysis and Machine Inteligence*, vol. 9, 1987, pp 698-700.

[4] B. K. P. Horn, Closed-form solution of absolute orientation using unit quaternions, *Optical Society of America J.*, vol. 5, no. 7, 1988, pp 1127-1135.

[5] B. K. P. Horn, and H. M. Hilden, and S. Negahdaripour, Closed-form solution of absolute orientation using orthonormal matrices, *Optical Society of America J.*, vol. 4, no. 7, 1987, pp 629-642.

[6] M. W. Walker, and L. Shao, and R. A. Volz, Estimating 3-D location parameters using dual quaternioins, *CVGIP: Image Undrestanding*, vol. 54, no. 3, 1991, pp 358-367.

[7] A. Lorusso, and D. W. Eggert, and R. B. Fisher, "A comparison of four algorithms for estimating 3-D rigid transformations", *British Machine Vision Conference*, Birmingham, England, 1995, pp. A20.

[8] X. Pennec and J P. A frame work for uncertaity and validation of 3D registration methods based on points and frames, *Comp. Vis. J.*, vol. 25, no. 3, 1997, pp 203-229.

[9] E. A. Wan, and R. Van Der Merwe, "The unscented Kalman filter for nonlinear estimation", *Adaptive Systems for Signal Processing, Communications, and Control Symposium*, 2000, pp.153-158.

[10] Simon. J. Julier and J. K. Uhlmann, Unscented Filtering and Nonlinear Estimation, *Proceeding of the IEEE*, vol. 92, no. 3, 2004, pp. 01-422.

[11] J. M. Fitzpatrick, and J. B. West, The distribution of target registration error in rigid-body point-based registration, *IEEE Transactions on Medical Imaging*, vol. 20, no. 9, 2004, pp. 917–927.

[12] S. J. Julier, and J. K. Uhlmann, "The Scaled Unscented Transformation", *Proceedings of the IEEE American Control Conference*, Anchorage AK, USA, 2002, pp. 4555-4559.